

CONVOLUTIONS WITH PROBABILITY
DISTRIBUTIONS, ZEROS OF L -FUNCTIONS,
AND THE LEAST QUADRATIC NONRESIDUE

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Abstract

Let \mathbf{d} be the density of a probability distribution that is compactly supported in the positive semi-axis. Under certain mild conditions we show that

$$\lim_{x \rightarrow \infty} x \sum_{n=1}^{\infty} \frac{\mathbf{d}^{*n}(x)}{n} = 1, \quad \text{where} \quad \mathbf{d}^{*n} := \underbrace{\mathbf{d} * \mathbf{d} * \cdots * \mathbf{d}}_{n \text{ times}}.$$

We also show that if $c > 0$ is a given constant for which the function $f(k) := \widehat{\mathbf{d}}(k) - 1$ does not vanish on the line $\{k \in \mathbb{C} : \Im k = -c\}$, where $\widehat{\mathbf{d}}$ is the Fourier transform of \mathbf{d} , then one has the asymptotic expansion

$$\sum_{n=1}^{\infty} \frac{\mathbf{d}^{*n}(x)}{n} = \frac{1}{x} \left(1 + \sum_k m(k) e^{-ikx} + O(e^{-cx}) \right) \quad (x \rightarrow +\infty),$$

where the sum is taken over those zeros k of f that lie in the strip $\{k \in \mathbb{C} : -c < \Im k < 0\}$, $m(k)$ is the multiplicity of any such zero, and the implied constant depends only on c . For a given distribution of this type, we briefly describe the location of the zeros k of f in the lower half-plane $\{k \in \mathbb{C} : \Im k < 0\}$.

For an odd prime p , let $n_0(p)$ be the least natural number such that $(n|p) = -1$, where $(\cdot|p)$ is the Legendre symbol. As an application of our work on probability distributions, we generalize a well known result of Heath-Brown concerning the exhibited behavior of the Dirichlet L -function $L(s, (\cdot|p))$ under the assumption that the Burgess bound $n_0(p) \ll p^{1/(4\sqrt{e})+\varepsilon}$ cannot be improved.

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1 Statement of results

In this paper, we establish a very general theorem concerning convolutions of certain compactly supported probability distributions. As an application to analytic number theory, we use our theorem to generalize a well known result of Heath-Brown concerning the behavior of the Dirichlet L -function attached to the Legendre symbol under an assumption that the Burgess bound on the least quadratic nonresidue cannot be improved.

1.1 Convolutions with probability distributions

Let \mathbf{d} be the density of a probability distribution that is supported in a finite interval $[a, b]$ with $a > 0$. Assume that \mathbf{d} is twice continuously differentiable on (a, b) , and that $\mathbf{d}(a)\mathbf{d}(b) \neq 0$. Put

$$F_{\mathbf{d}}(x) := \sum_{n=1}^{\infty} \frac{\mathbf{d}^{*n}(x)}{n} \quad (x > 0), \quad (1.1)$$

where \mathbf{d}^{*n} denotes the n -fold convolution of \mathbf{d} with itself, i.e.,

$$\mathbf{d}^{*n} := \underbrace{\mathbf{d} * \mathbf{d} * \cdots * \mathbf{d}}_{n \text{ times}} \quad (n \in \mathbb{N}).$$

Since $\mathbf{d}^{*n}(x) = 0$ whenever $x < na$, for every $x > 0$ the series (1.1) has only finitely many nonzero terms, hence the function $F_{\mathbf{d}}$ is well-defined pointwise (but not absolutely summable; see Corollary 2.4).

In this paper, we show that the leading term in the asymptotic expansion of $F_{\mathbf{d}}(x)$ as $x \rightarrow \infty$ is universal, i.e., it does not depend on the particular choice of \mathbf{d} for a wide class of distributions, while the (exponentially small) higher order terms of the asymptotics are determined by the roots in the lower half-plane of the equation

$$\widehat{\mathbf{d}}(k) = 1.$$

Here $\widehat{\mathbf{d}}$ denotes the Fourier transform of \mathbf{d} , which is an entire function given by

$$\widehat{\mathbf{d}}(k) := \int_a^b \mathbf{d}(x) e^{ikx} dx \quad (k \in \mathbb{C}).$$

Theorem 1.1. *For any constant $c > 0$, the function $f(k) := \widehat{\mathbf{d}}(k) - 1$ has only finitely many zeros in the strip $\Pi_c := \{k \in \mathbb{C} : -c < \Im k < 0\}$. Suppose that f does not vanish on the line $\{k \in \mathbb{C} : \Im k = -c\}$. Then*

$$F_{\mathbf{d}}(x) = \frac{1}{x} \left(1 + \sum_k m(k) e^{-ikx} + E(c, x) e^{-cx} \right) \quad (x > 0), \quad (1.2)$$

where the sum is taken over those zeros k of f that lie in Π_c , $m(k)$ is the multiplicity of any such zero, and

$$E(c, x) := \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{\widehat{\mathbf{d}}'(u - ic)}{1 - \widehat{\mathbf{d}}(u - ic)} - \widehat{\mathbf{d}}'(u - ic) \right) e^{-iux} du. \quad (1.3)$$

Remarks. Note that for any $c > 0$ satisfying the hypotheses of Theorem 1.1, the quantity $E(c, x)$ satisfies the uniform bound

$$|E(c, x)| \leq \frac{1}{2\pi} \|f_c\|_{L^1(\mathbb{R})} \quad (x > 0),$$

where

$$f_c(u) := \frac{\widehat{\mathbf{d}}'(u - ic)}{1 - \widehat{\mathbf{d}}(u - ic)} - \widehat{\mathbf{d}}'(u - ic) \quad (u \in \mathbb{R}).$$

The proof of Theorem 1.1 is given in §2, and in §3 we briefly explore the location of the zeros of the function $f(k) := \widehat{\mathbf{d}}(k) - 1$ that lie in the lower half-plane.

1.2 The least quadratic nonresidue

For any odd prime p , let $n_0(p)$ denote the least positive quadratic nonresidue modulo p ; that is,

$$n_0(p) := \min\{n \in \mathbb{N} : (n|p) = -1\},$$

where $(\cdot|p)$ is the *Legendre symbol*. The first nontrivial bound on $n_0(p)$ was given by Gauss [6, Article 129], who showed that $n_0(p) < 2\sqrt{p} + 1$ holds for every prime $p \equiv 1 \pmod{8}$. Vinogradov [20] proved that $n_0(p) \ll p^\kappa$ holds for all primes p provided that $\kappa > 1/(2\sqrt{e})$, and later, Burgess [2] extended

this range to include all real numbers $\kappa > 1/(4\sqrt{e})$. The latter result has not been improved since 1957.

An old conjecture of Vinogradov asserts that the bound $n_0(p) \ll p^\varepsilon$ holds for every fixed $\varepsilon > 0$. Linnik [13] showed that Vinogradov's conjecture is true under the *Extended Riemann Hypothesis* (ERH). A decade later, Ankeny [1] proved that the stronger bound $n_0(p) \ll (\log p)^2$ holds under the ERH.

It is natural to wonder what bounds on $n_0(p)$ can be established under weaker conditional hypotheses than the ERH. The pioneering work in this direction (which largely motivates the present paper) is an unpublished analysis of Heath-Brown concerning the behavior of the Dirichlet L -function $L(s, (\cdot|p))$ under an assumption that the Burgess bound is *tight*, i.e., that the lower bound $n_0(p) \geq p^{1/(4\sqrt{e})}$ holds for infinitely many primes p ; we refer the reader to Diamond *et al* [4, Appendix] for a superb account of Heath-Brown's methods and results.

In this paper, we modify and extend Heath-Brown's ideas as follows. Throughout, let κ, λ be fixed real numbers such that

$$0 < \kappa \leq \frac{\lambda}{\sqrt{e}} \leq \frac{1}{4\sqrt{e}}. \quad (1.4)$$

For every odd prime p , put

$$\mathcal{N}_p(X) := \{n \leq X : (n|p) = -1\} \quad (X > 0). \quad (1.5)$$

We assume that there is an infinite set of primes \mathcal{P} for which

$$n_0(p) \geq p^\kappa \quad (p \in \mathcal{P}). \quad (1.6)$$

Our aim is to understand how the zeros of $L(s, (\cdot|p))$ are constrained by the condition (1.6) (as previously mentioned, such a set \mathcal{P} cannot exist under the ERH by the work of Linnik [13]).

In addition to (1.6) we also assume that for any fixed $\theta \geq 0$ the estimate

$$|\mathcal{N}_p(p^\theta)| = (\delta(\theta) + \underline{o}(1))p^\theta \quad (1.7)$$

holds, where $\underline{o}(1)$ denotes an error term that tends to zero as $p \rightarrow \infty$ with primes p lying in the set \mathcal{P} , and δ is a function of the form

$$\delta(\theta) := \frac{1}{2} \int_0^\theta \mathrm{d}(u) \, du \quad (\theta \geq 0) \quad (1.8)$$

with some probability distribution \mathbf{d} that is supported in the interval $[\kappa, \lambda]$ and twice continuously differentiable on (κ, λ) , with $\mathbf{d}(\kappa)\mathbf{d}(\lambda) \neq 0$.

We remark that the inequalities in (1.4) are optimal in a certain sense (see Lemmas 4.5 and Theorem 5.1 below). The main result of the paper is as follows.

Theorem 1.2. *Under the hypotheses (1.6), (1.7) and (1.8), for every nonzero root k of the equation $\widehat{\mathbf{d}}(k) = 1$ there is a complex sequence $(\varrho_p)_{p \in \mathcal{P}}$ with $L(\varrho_p, (\cdot|p)) = 0$ such that $(\varrho_p - 1) \log p \rightarrow -ik$ as $p \rightarrow \infty$ with $p \in \mathcal{P}$.*

Remarks. In the special case that $\kappa := 1/(4\sqrt{e})$ and $\lambda := 1/4$, we show in §5 that under hypothesis (1.6) the condition (1.7) is automatically met with the function δ given by

$$\delta(\theta) := \begin{cases} 0 & \text{if } 0 \leq \theta \leq 1/(4\sqrt{e}), \\ \log(4\theta\sqrt{e}) & \text{if } 1/(4\sqrt{e}) \leq \theta \leq 1/4, \\ 1/2 & \text{if } \theta \geq 1/4, \end{cases} \quad (1.9)$$

and the probability distribution \mathbf{d} defined by

$$\mathbf{d}(x) := \begin{cases} 2x^{-1} & \text{if } 1/(4\sqrt{e}) \leq x \leq 1/4, \\ 0 & \text{otherwise.} \end{cases} \quad (1.10)$$

Then, from the conclusion of Theorem 1.2 we recover the aforementioned result of Heath-Brown. We also note that in any application of Theorem 1.2 it is useful to have information about the location of the zeros of the function $f(k) := \widehat{\mathbf{d}}(k) - 1$. General results of this nature are given in Proposition 3.1, where we outline a standard method for obtaining such information.

We also remark that the normalization factor $\frac{1}{2}$ in hypothesis (1.8) is chosen to meet the unconditional requirement that

$$\lim_{p \rightarrow \infty} \frac{|\mathcal{N}_p(p^\theta)|}{p^\theta} = \frac{1}{2} \quad (\theta \geq 1/4);$$

see Lemma 4.5.

The proof of Theorem 1.2 (see §4 below) can be summarized as follows. First, we show that the limit

$$S_1(\theta) := \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{q \leq p^\theta \\ (q|p) = -1}} q^{-1} \quad (1.11)$$

exists for all $\theta \geq 0$, where the sum is taken over prime nonresidues $q \leq p^\theta$. Using properties of the Laplace transform we show that S_1 is continuously differentiable on (λ, ∞) and that

$$S'_1(\theta) = \frac{1}{2} \sum_{n \in \mathbb{N}} \frac{\mathbf{d}^{*n}(\theta)}{n} \quad (\theta > \lambda), \quad (1.12)$$

where \mathbf{d}^{*n} denotes the n -fold convolution $\mathbf{d} * \cdots * \mathbf{d}$ as before. Taking into account Theorem 1.1, for any fixed $c > 0$ we obtain an estimate of the form

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_k m(k) e^{-ik\theta} + \frac{E(c, \theta) e^{-c\theta}}{2\theta},$$

where $E(c, \theta)$ is given by (1.3). On the other hand, expressing the derivative $S'_1(\theta)$ as a limit of difference quotients and using standard estimates from number theory, we derive that for any fixed $c > 0$ one has

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\varrho} \frac{\tilde{m}(\varrho)}{\varrho p^{(1-\varrho)\theta}} + O(e^{-c\theta}),$$

where each sum runs over the distinct zeros $\varrho = \beta + i\gamma$ of $L(s, (\cdot|p))$ in the region determined by the inequalities

$$\beta > 1 - c/\log p \quad \text{and} \quad |\gamma| \leq p,$$

and $\tilde{m}(\varrho)$ is the multiplicity of any such zero. A comparison of these two relations leads to the statement of Theorem 1.2.

Not too surprisingly, our proof of Theorem 1.2 incorporates principles that figure prominently in treatments of Linnik's Theorem, including the log-free zero-density estimate (see Linnik [14]) and the Deuring-Heilbronn phenomenon (see Linnik [15]). On the other hand, our method of applying the Laplace transform to derive (1.12) appears to be new.

2 Proof of Theorem 1.1

We continue to assume that \mathbf{d} has the properties listed in §1.1; that is, the function \mathbf{d} is twice continuously differentiable on (a, b) , and $\mathbf{d}(a)\mathbf{d}(b) \neq 0$.

Lemma 2.1. *We have*

$$\widehat{\mathbf{d}}(k) = 1 + id_1 k - \frac{1}{2}d_2 k^2 + O(k^3) \quad (k \rightarrow 0),$$

where

$$d_1 := \int_a^b x \, \mathbf{d}(x) \, dx \quad \text{and} \quad d_2 := \int_a^b x^2 \, \mathbf{d}(x) \, dx.$$

Also,

$$\widehat{\mathbf{d}}(k) = \frac{1}{ik} (\mathbf{d}(b)e^{ikb} - \mathbf{d}(a)e^{ika}) + O(k^{-2}) \quad (k \rightarrow \infty).$$

Proof. The first representation follows by expanding e^{ikx} as a power series around $k = 0$, whereas the second is obtained using integration by parts

$$\begin{aligned} \widehat{\mathbf{d}}(k) &= \int_a^b \mathbf{d}(x) e^{ikx} \, dx = \frac{1}{ik} (\mathbf{d}(b)e^{ikb} - \mathbf{d}(a)e^{ika}) + \frac{1}{k^2} (\mathbf{d}'(b)e^{ikb} - \mathbf{d}'(a)e^{ika}) \\ &\quad - \frac{1}{k^2} \int_a^b \mathbf{d}''(x) e^{ikx} \, dx \end{aligned}$$

together with the fact that $\mathbf{d}''(x)$ is a continuous function on $[a, b]$. \square

Remark. In the lower half-plane we have the estimate

$$\widehat{\mathbf{d}}(k) = O\left(\frac{e^{|\Im k|b}}{|k|}\right),$$

which holds uniformly with respect to $\arg k$.

Lemma 2.2. $F_{\mathbf{d}} \in L^2(\mathbb{R})$.

Proof. It is enough to prove that the series (1.1) converges in $L^2(\mathbb{R})$. In turn, since $\widehat{\mathbf{d}}^n$ is the Fourier transform of \mathbf{d}^{*n} for each $n \in \mathbb{N}$, it suffices to show that the series

$$\sum_{n=1}^{\infty} \frac{\widehat{\mathbf{d}}(k)^n}{n} \quad (k \in \mathbb{R}) \tag{2.1}$$

converges in $L^2(\mathbb{R})$.

Step 1. First we note that

$$|\widehat{\mathbf{d}}(k)| = \left| \int_a^b \mathbf{d}(x) e^{ikx} dx \right| < \int_a^b \mathbf{d}(x) dx = 1 \quad (k \in \mathbb{R} \setminus \{0\}) \quad (2.2)$$

since \mathbf{d} is nonnegative and not identically zero, hence the series (2.1) converges uniformly on every compact set $\Omega \subset \mathbb{R} \setminus \{0\}$; this proves, in particular, that the series (2.1) converges in $L^2(\Omega)$.

Step 2. By Lemma 2.1 it is easy to see that there exists $\delta > 0$ such that

$$|\widehat{\mathbf{d}}(k)| \leq \sqrt{1 - C_1 k^2} \quad (k \in [-\delta, \delta])$$

holds for some positive constant C_1 that is less than

$$d_2 - d_1^2 = \int_a^b x^2 \mathbf{d}(x) dx - \left(\int_a^b x \mathbf{d}(x) dx \right)^2 > 0.$$

Using Laplace's method (see, e.g., [17, Ch. 3, Sec. 7]), we find that

$$\int_{-\delta}^{\delta} |\widehat{\mathbf{d}}(k)|^n dk \leq \int_{-\delta}^{\delta} (1 - C_1 k^2)^{n/2} dk = \left(\frac{2\pi}{C_1 n} \right)^{1/2} + O(n^{-3/2});$$

that is, for δ chosen as above, the inequality

$$\int_{-\delta}^{\delta} |\widehat{\mathbf{d}}(k)|^n dk \leq \frac{C_2}{\sqrt{n}} \quad (n \in \mathbb{N})$$

holds for some constant $C_2 > 0$. Hence,

$$\|\widehat{\mathbf{d}}^n\|_{L^2(-\delta, \delta)} \leq \frac{\sqrt{C_2}}{(2n)^{1/4}}. \quad (2.3)$$

Now, for any natural numbers $M > N$, from (2.3) we deduce that

$$\left\| \sum_{n=N}^M \frac{\widehat{\mathbf{d}}^n}{n} \right\|_{L^2(-\delta, \delta)} \leq \sum_{n=N}^M \frac{1}{n} \|\widehat{\mathbf{d}}^n\|_{L^2(-\delta, \delta)} \leq \frac{\sqrt{C_2}}{2^{1/4}} \sum_{n=N}^M \frac{1}{n^{5/4}},$$

which shows that the series (2.1) converges in $L^2(-\delta, \delta)$.

Step 3. By Lemma 2.1 it is also clear that there is a constant A such that for every sufficiently large $R > 0$ the inequality

$$|\widehat{\mathbf{d}}(k)| \leq \frac{A}{|k|} \quad (|k| > R).$$

holds. Increasing R if necessary, we can assume that $R > A$; then, for any natural numbers $M > N \geq 2$ we have

$$\begin{aligned} \left\| \sum_{n=N}^M \frac{\widehat{\mathbf{d}}^n}{n} \right\|_{L^2(\mathbb{R} \setminus [-R, R])} &\leq \sum_{n=N}^M \frac{1}{n} \left(A^{2n} \int_{|k|>R} \frac{dk}{|k|^{2n}} \right)^{1/2} \\ &\leq \sqrt{2R} \sum_{n=N}^M \frac{1}{n\sqrt{2n-1}} \left(\frac{A}{R} \right)^n \\ &\leq \sqrt{2R} \sum_{n=N}^M \frac{1}{n\sqrt{2n-1}}, \end{aligned}$$

which shows that the series (2.1) converges in $L^2(\mathbb{R} \setminus [-R, R])$.

Combining the results of the steps above, we conclude that the series (2.1) converges in $L^2(\mathbb{R})$ as required, and the lemma is proved. \square

Lemma 2.3. *We have*

$$F_{\mathbf{d}}(x) - \mathbf{d}(x) = -\frac{1}{2\pi} \int_{\mathbb{R}} (\log(1 - \widehat{\mathbf{d}}(k)) + \widehat{\mathbf{d}}(k)) e^{-ikx} dk \quad (x \in \mathbb{R}). \quad (2.4)$$

Proof. From the bound (2.2) we see that

$$\widehat{F}_{\mathbf{d}}(k) = \sum_{n=1}^{\infty} \frac{\widehat{\mathbf{d}}(k)^n}{n} = -\log(1 - \widehat{\mathbf{d}}(k)) \quad (k \in \mathbb{R} \setminus \{0\}).$$

Since $F_{\mathbf{d}} \in L^2(\mathbb{R})$ by Lemma 2.2, one obtains that

$$F_{\mathbf{d}}(x) = -\frac{1}{2\pi} \text{l.i.m.} \int_{\mathbb{R}} \log(1 - \widehat{\mathbf{d}}(k)) e^{-ikx} dk \quad (\text{a.e. } x \in \mathbb{R}).$$

Therefore,

$$F_{\mathbf{d}}(x) - \mathbf{d}(x) = -\frac{1}{2\pi} \text{l.i.m.} \int_{\mathbb{R}} (\log(1 - \widehat{\mathbf{d}}(k)) + \widehat{\mathbf{d}}(k)) e^{-ikx} dk \quad (2.5)$$

for almost all $x \in \mathbb{R}$.

However, since the integrand $\log(1 - \widehat{\mathbf{d}}) + \widehat{\mathbf{d}}$ is absolutely summable by Lemma 2.1, we see that (2.5) implies (2.4) since the functions on either side of (2.4) are continuous, and every L^2 -function (i.e., equivalence class of functions) has at most one continuous representative. \square

Proof of Theorem 1.1. Since $\mathbf{d}(x) = 0$ for $x > b$, from Lemma 2.3 it follows that

$$xF_{\mathbf{d}}(x) = -\frac{x}{2\pi} \int_{\mathbb{R}} (\log(1 - \widehat{\mathbf{d}}(k)) + \widehat{\mathbf{d}}(k)) e^{-ikx} dk \quad (x > b).$$

Recall that $\log(1 - \widehat{\mathbf{d}}) + \widehat{\mathbf{d}} \in L^1(\mathbb{R})$, and therefore

$$\begin{aligned} \int_{\mathbb{R}} (\log(1 - \widehat{\mathbf{d}}(k)) + \widehat{\mathbf{d}}(k)) e^{-ikx} dk &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} (\log(1 - \widehat{\mathbf{d}}(k)) + \widehat{\mathbf{d}}(k)) e^{-ikx} dk \\ &= \text{V.P.} \int_{\mathbb{R}} (\log(1 - \widehat{\mathbf{d}}(k)) + \widehat{\mathbf{d}}(k)) e^{-ikx} dk. \end{aligned}$$

Integration by parts yields the relation

$$xF_{\mathbf{d}}(x) = \frac{1}{2} + \frac{1}{2\pi i} \text{V.P.} \int_{\mathbb{R}} \left(\frac{\widehat{\mathbf{d}}'(k)}{1 - \widehat{\mathbf{d}}(k)} - \widehat{\mathbf{d}}'(k) \right) e^{-ikx} dk \quad (x > b) \quad (2.6)$$

as Lemma 2.1 implies that

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0^+} ([\log(1 - \widehat{\mathbf{d}}(-\varepsilon)) + \widehat{\mathbf{d}}(-\varepsilon)] e^{i\varepsilon x} - [\log(1 - \widehat{\mathbf{d}}(\varepsilon)) + \widehat{\mathbf{d}}(\varepsilon)] e^{-i\varepsilon x}) \\ &= \lim_{\varepsilon \rightarrow 0^+} (\log(1 - \widehat{\mathbf{d}}(-\varepsilon)) e^{i\varepsilon x} - \log(1 - \widehat{\mathbf{d}}(\varepsilon)) e^{-i\varepsilon x}) \\ &= \lim_{\varepsilon \rightarrow 0^+} (\log(id_1 \varepsilon) - \log(-id_1 \varepsilon)) = \pi i, \end{aligned}$$

with

$$d_1 = \int_a^b x \mathbf{d}(x) dx > 0,$$

the mean of the distribution \mathbf{d} .

Next, concerning the V.P. integral in (2.6) we have

$$\begin{aligned} \text{V.P.} \int_{\mathbb{R}} \left(\frac{\widehat{\mathbf{d}}'(k)}{1 - \widehat{\mathbf{d}}(k)} - \widehat{\mathbf{d}}'(k) \right) e^{-ikx} dk &:= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \left(\frac{\widehat{\mathbf{d}}'(k)}{1 - \widehat{\mathbf{d}}(k)} - \widehat{\mathbf{d}}'(k) \right) e^{-ikx} dk \\ &= \lim_{\varepsilon \rightarrow 0^+} (\mathcal{I}_{\varepsilon} - \mathcal{I}'_{\varepsilon}), \end{aligned}$$

where

$$\begin{aligned}\mathcal{I}'_\varepsilon &:= \int_{\Gamma'_\varepsilon} \left(\frac{\widehat{\mathbf{d}}'(z)}{1 - \widehat{\mathbf{d}}(z)} - \widehat{\mathbf{d}}'(z) \right) e^{-izx} dz \quad \text{with } \Gamma'_\varepsilon := \{z \in \mathbb{C} : |z| = \varepsilon, \Im z < 0\}, \\ \mathcal{I}_\varepsilon &:= \int_{\Gamma_\varepsilon} \left(\frac{\widehat{\mathbf{d}}'(z)}{1 - \widehat{\mathbf{d}}(z)} - \widehat{\mathbf{d}}'(z) \right) e^{-izx} dz \quad \text{with } \Gamma_\varepsilon := (-\infty, -\varepsilon) \cup \Gamma'_\varepsilon \cup (\varepsilon, \infty).\end{aligned}$$

Here, Γ'_ε and Γ_ε are oriented so that $\Re z$ is increasing on each contour. Using Lemma 2.1 again, it is easy to see that

$$\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \mathcal{I}'_\varepsilon = \frac{1}{2} \operatorname{Res} \Big|_{z=0} \left(\frac{\widehat{\mathbf{d}}'(z)}{1 - \widehat{\mathbf{d}}(z)} - \widehat{\mathbf{d}}'(z) \right) e^{-izx} = -\frac{1}{2}.$$

Using this information in (2.6) and applying the Residue Theorem, we have

$$\begin{aligned}xF_{\mathbf{d}}(x) &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \mathcal{I}_\varepsilon \\ &= 1 + \frac{1}{2\pi i} \int_{\Im k = -c} \left(\frac{\widehat{\mathbf{d}}'(k)}{1 - \widehat{\mathbf{d}}(k)} - \widehat{\mathbf{d}}'(k) \right) e^{-ikx} dk \\ &\quad - \sum_k \operatorname{Res} \Big|_{z=k} \left(\frac{\widehat{\mathbf{d}}'(z)}{1 - \widehat{\mathbf{d}}(z)} - \widehat{\mathbf{d}}'(z) \right) e^{-izx} + E_1 + E_2,\end{aligned}$$

where the integral over $\{k \in \mathbb{C} : \Im k = -c\}$ is oriented with $\Re k$ increasing, the sum is taken over all roots k of the equation $\widehat{\mathbf{d}}(k) = 1$ for which $k \in \Pi_c$, and

$$\begin{aligned}E_1 &:= -\frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_0^c \left(\frac{\widehat{\mathbf{d}}'(-R - iu)}{1 - \widehat{\mathbf{d}}(-R - iu)} - \widehat{\mathbf{d}}'(-R - iu) \right) e^{-ux + iR} du, \\ E_2 &:= \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{-c}^0 \left(\frac{\widehat{\mathbf{d}}'(R + iu)}{1 - \widehat{\mathbf{d}}(R + iu)} - \widehat{\mathbf{d}}'(R + iu) \right) e^{ux - iR} du.\end{aligned}$$

Denoting by $m(k)$ the multiplicity of each root k in the sum, we have

$$\operatorname{Res} \Big|_{z=k} \left(\frac{\widehat{\mathbf{d}}'(z)}{1 - \widehat{\mathbf{d}}(z)} - \widehat{\mathbf{d}}'(z) \right) e^{-izx} = -m(k) e^{-ikx}.$$

Taking into account that $E_1 = E_2 = 0$ by Remark 2, we finish the proof. \square

Corollary 2.4. $F_{\mathbf{d}} \in (L^1_{\mathbf{w}}(\mathbb{R}_+) \setminus L^1(\mathbb{R}_+)) \cap L^2(\mathbb{R}_+).$

Proof. The fact that $F_{\mathbf{d}} \notin L^1(\mathbb{R}_+)$ follows from the definition (1.1) and the observations that

$$\int_0^\infty \mathbf{d}^{*n}(x) dx = 1$$

and that the series $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

The membership $F_{\mathbf{d}} \in L^2(\mathbb{R}_+)$ is the content of Lemma 2.2.

Finally, $F_{\mathbf{d}}$ belongs to the weak space $L^1_{\mathbf{w}}(\mathbb{R}_+)$ since $F_{\mathbf{d}}$ is a bounded function and it admits the estimate

$$F_{\mathbf{d}}(x) = O(x^{-1}) \quad (x \rightarrow \infty)$$

(this follows from the asymptotics (1.2)), hence

$$\sup_{t>0} t \cdot \text{mes}\{x > 0 : |F(x)| > t\} < \infty.$$

This completes the proof. \square

3 On solutions to the equation $\widehat{\mathbf{d}}(k) = 1$

In this section we briefly describe the location of zeros of the function $\widehat{\mathbf{d}}(k) - 1$ that lie in the lower half-plane. Our results here concerning the distribution of the zeros are not used in the proof of Theorem 1.2 in §4 below. However, as Theorem 1.2 shows, these zeros suitably translated and rescaled are zeros of the L -function.

In view of the remark following Lemma 2.1 we see that the aforementioned zeros lie asymptotically close to solutions of the equation

$$e^{ikb} = \frac{ik}{\mathbf{d}(b)} \quad (\Im k < 0). \quad (3.1)$$

The solutions to (3.1) can be determined explicitly in terms of the Lambert W -function or estimated using standard methods going back to Horn [10, 11] (see also Hardy [8], Zdanovich [22], Pavlov [18, 19], and Zworski [23]).

Proposition 3.1. (cf. [4, Lemma 2]) *The zeros of the equation $\widehat{\mathbf{d}}(k) = 1$ satisfy the asymptotic formula*

$$k_{\pm n} = \pm \frac{\pi}{b} (2n + \tfrac{1}{2}) - \frac{i}{b} \log \frac{2\pi n}{b \mathbf{d}(b)} + o(1) \quad (n \in \mathbb{N}, n \rightarrow \infty).$$

A heuristic argument proceeds as follows. To find solutions to (3.1) we introduce a new variable $z = ikb$ and rewrite (3.1) in the form

$$e^z = \alpha z \quad \text{with} \quad \alpha := (b \mathfrak{d}(b))^{-1}.$$

We prepare this equation for “bootstrapping” by writing it in the form

$$z = \log(\alpha z) + 2\pi in$$

with a fixed $n \in \mathbb{N}$. We apply the Banach fixed point theorem, starting the iterative process with

$$\begin{aligned} z^{(0)} &:= 2\pi in, \\ z^{(1)} &:= \log(\alpha z^{(0)}) + 2\pi in = \log(2\pi n \alpha) + \pi i(2n + \tfrac{1}{2}), \end{aligned}$$

and continuing in this way by putting

$$z_n^{(j+1)} := \log(\alpha z_n^{(j)}) + 2\pi in \quad (j \geq 2).$$

If n and j are large we see that

$$z_n^{(j)} = \log(2\pi n \alpha) + \pi i(2n + \tfrac{1}{2}) + (\text{lower order terms}).$$

Returning to the original variable k we conclude that the zeros k_n with $\Re k_n > 0$ and $\Im k_n < 0$ satisfy

$$k_n \sim \frac{\pi}{b}(2n + \tfrac{1}{2}) - \frac{i}{b} \log(2\pi n \alpha) + o(1) \quad (n \rightarrow \infty).$$

The heuristic argument is completed by noting that the zeros of $\widehat{\mathfrak{d}}(\zeta) - 1$ are located symmetrically with respect to the imaginary axis.

4 Proof of Theorem 1.2

4.1 Some technical lemmas

For the proof of Theorem 1.2 we need several technical results.

For any Dirichlet character χ we denote by $N(\sigma, T, \chi)$ the number of zeros of $L(s, \chi)$ in the region $\{s \in \mathbb{C} : \sigma \leq \Re s \leq 1, |\Im s| \leq T\}$, counted with multiplicity. The following “log-free” zero-density estimate is due to Linnik [14].

Lemma 4.1. *There is an effectively computable constant $c_1 > 0$ such that the bound*

$$\sum_{\chi \pmod{q}} N(\sigma, T, \chi) \ll (qT)^{c_1(1-\sigma)}$$

holds uniformly for $q \geq 1$, $\sigma \in [0, 1]$ and $T \geq 1$.

For our proof of Theorem 1.2, putative Siegel zeros have an impact, and exceptional moduli must be taken into account; see Davenport [3, §14] for a general background on exceptional moduli. For the purposes of this paper, we need only the following specialized result, which is a quantitative version of the Deuring-Heilbronn phenomenon (see Linnik [15]); for a more general statement, we refer the reader to Davenport [3, §§13–14] and Knapowski [12] (see also Gallagher [5]).

Lemma 4.2. *There exist positive constants c_2, c_3 with the following property. Let χ be a primitive Dirichlet character modulo q , where $q > 1$. Then $L(s, \chi)$ has at most one zero $\varrho = \beta + i\gamma$ such that*

$$\beta > 1 - \frac{c_2}{\log q} \quad \text{and} \quad |\gamma| \leq q.$$

If there is such an exception, then the exceptional zero is real, simple and unique. Moreover, denoting by β_1 the exceptional zero, we have $L(s, \chi) \neq 0$ if $s = \sigma + it \neq \beta_1$ satisfies

$$\sigma > 1 - \frac{c_3}{\log q} \log \left(\frac{ec_2}{(1 - \beta_1) \log q} \right) \quad \text{and} \quad |t| \leq q.$$

The next result, which may be of independent interest, is a variant of Montgomery and Vaughan [16, Exercise 2, p. 382]; our proof uses ideas of Gallagher (see [5, §4]).

Lemma 4.3. *There is an effectively computable constant $c_4 > 0$ with the following property. Let χ be a primitive Dirichlet character modulo q , where $q > 1$, and put*

$$\psi(x, \chi) := \sum_{n \leq x} \chi(n) \Lambda(n) \quad (x > 0),$$

where Λ is the von Mangoldt function. For any $c > 0$ there is a constant $K = K(c)$ such that the estimate

$$\psi(x, \chi) = - \sum_{\varrho} \tilde{m}(\varrho) \frac{x^{\varrho}}{\varrho} + O \left(x \exp \left(-c \frac{\log x}{\log q} \right) \right)$$

holds uniformly provided that

$$\exp(K\sqrt{\log x}) \leq q \leq x^{c_4}, \quad (4.1)$$

where the sum is taken over distinct zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ for which $\beta > 1 - 2c/\log q$ and $|\gamma| \leq q$, $\tilde{m}(\varrho)$ is the multiplicity of any such zero, and the implied constant depends only on c .

Proof. Let $c_4 := \min\{\frac{3}{4}, (4c_1)^{-1}\}$, where c_1 is the constant described in Lemma 4.1.

We have by Davenport [3, §19] (with $T := q$):

$$\psi(x, \chi) = - \sum_{\varrho \in \mathcal{Z}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} + R(x, q), \quad (4.2)$$

where

$$|R(x, q)| \ll xq^{-1} \log^2 qx + x^{1/4} \log x$$

and \mathcal{Z} denotes the set of nontrivial zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ such that $0 \leq \beta \leq 1$ and $|\gamma| \leq q$. (Recall that if $\chi(-1) = 1$, the Dirichlet L -function $L(s, \chi)$ vanishes at $s = 0$; however, this trivial zero $\varrho = 0$ is not included in the sum).

Since by hypothesis $q \leq x^{c_4}$ and $c_4 \leq \frac{3}{4}$, we have

$$\log^2 qx = O(\log^2 x) \quad \text{and} \quad x^{1/4} \log x = O\left(\frac{x \log^2 x}{q}\right),$$

and therefore

$$|R(x, q)| = O\left(\frac{x \log^2 x}{q}\right),$$

which together with (4.2) proves that

$$\psi(x, \chi) = - \sum_{\varrho \in \mathcal{Z}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} + O\left(\frac{x \log^2 x}{q}\right) \quad (x \rightarrow \infty) \quad (4.3)$$

(or $q \rightarrow \infty$, cf. (4.1)).

We observe that for any fixed $K > \sqrt{c}$ we have the following estimate

$$\frac{x \log^2 x}{q} = O(E) \quad (q \geq \exp(K\sqrt{\log x})),$$

where

$$E := x \exp(-c(\log x)/\log q).$$

Thus (4.3) yields the representation

$$\psi(x, \chi) = - \sum_{\varrho \in \mathcal{Z}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} + O(E).$$

Put $\eta := 2c/\log q$, and let \mathcal{Z}_1 and \mathcal{Z}_2 be the set of zeros in \mathcal{Z} that satisfy $\beta \leq 1 - \eta$ and $\beta > 1 - \eta$, respectively. To prove the lemma, we need to show that

$$\sum_{\varrho \in \mathcal{Z}_1} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \ll E. \quad (4.4)$$

To do this, choose some $\beta_0 \in (0, \frac{1}{2})$ and split \mathcal{Z} into disjoint subsets

$$\mathcal{Z} = \mathcal{L} \cup \mathcal{R},$$

where \mathcal{L} and \mathcal{R} denote the set of zeros in \mathcal{Z}_1 that satisfy the inequalities $0 < \beta \leq \beta_0$ and $\beta_0 < \beta \leq 1 - \eta$, respectively. Define the corresponding zero-counting functions

$$N_{\mathcal{L}}(\sigma, q, \chi) := N(\sigma, q, \chi) - N(\beta_0, q, \chi) \quad (\sigma \in [0, \beta_0])$$

and

$$N_{\mathcal{R}}(\sigma, q, \chi) := N(\sigma, q, \chi) - N(1 - \eta, q, \chi) \quad (\sigma \in [\beta_0, 1 - \eta]),$$

where as above $N(\sigma, q, \chi)$ denotes the number of zeros of $L(s, \chi)$ in the region

$$\{s \in \mathbb{C} : \sigma \leq \Re s \leq 1, |\Im s| \leq q\},$$

counted with multiplicity.

To bound $\sum_{\varrho \in \mathcal{R}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho}$ we begin by observing that

$$\begin{aligned} \min_{\varrho \in \mathcal{R}} |\varrho| \cdot \left| \sum_{\varrho \in \mathcal{R}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| &\ll \sum_{\varrho \in \mathcal{R}} \tilde{m}(\varrho) x^{\Re(\varrho)} = - \int_{\beta_0}^{1-\eta} x^\alpha dN_{\mathcal{R}}(\alpha, q, \chi) \\ &= x^{\beta_0} N_{\mathcal{R}}(\beta_0, q, \chi) + \log x \int_{\beta_0}^{1-\eta} x^\alpha N_{\mathcal{R}}(\alpha, q, \chi) d\alpha \\ &\leq x^{\beta_0} N(0, q, \chi) + \log x \int_0^{1-\eta} x^\alpha N(\alpha, q, \chi) d\alpha. \end{aligned}$$

Since $c_4 \leq (4c_1)^{-1}$ we have by Lemma 4.1:

$$N(\alpha, q, \chi) \ll q^{2c_1(1-\alpha)} \leq x^{1/2(1-\alpha)};$$

thus,

$$\begin{aligned} x^{\beta_0} N(0, q, \chi) + \log x \int_0^{1-\eta} x^\alpha N(\alpha, q, \chi) d\alpha &\ll x^{\beta_0+1/2} + \log x \int_0^{1-\eta} x^{\frac{1}{2}(1+\alpha)} d\alpha \\ &\ll x^{\beta_0+1/2} + x^{1-\eta/2}. \end{aligned}$$

Consequently, we have

$$\left| \sum_{\varrho \in \mathcal{R}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| \ll \frac{1}{\min_{\varrho \in \mathcal{R}} |\varrho|} (x^{\beta_0+1/2} + x^{1-\eta/2}).$$

Since

$$\min_{\varrho \in \mathcal{R}} |\varrho| \geq \frac{1}{\beta_0} > 2$$

we deduce that

$$\left| \sum_{\varrho \in \mathcal{R}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| \ll x^{1-\eta/2} = E = O(E). \quad (4.5)$$

To estimate the sum $\sum_{\varrho \in \mathcal{L}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho}$ we proceed in a similar way, assuming initially that the character χ is not exceptional (that is, the function $L(s, \chi)$ has no Siegel zero). We have

$$\begin{aligned} \min_{\varrho \in \mathcal{L}} |\varrho| \cdot \left| \sum_{\varrho \in \mathcal{L}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| &\ll \sum_{\varrho \in \mathcal{L}} \tilde{m}(\varrho) x^{\beta(\varrho)} = - \int_0^{\beta_0} x^\alpha dN_{\mathcal{L}}(\alpha, q, \chi) \\ &= N_{\mathcal{L}}(0, q, \chi) + \log x \int_0^{\beta_0} x^\alpha N_{\mathcal{L}}(\alpha, q, \chi) d\alpha \\ &\leq N(0, q, \chi) + \log x \int_0^{\beta_0} x^\alpha N(\alpha, q, \chi) d\alpha \\ &\ll x^{1/2} + x^{(1+\beta_0)/2} \ll x^{(1+\beta_0)/2}. \end{aligned}$$

Applying Lemma 4.2 and using the functional equation for the L -function, we have the lower bound

$$\min_{\varrho \in \mathcal{L}} |\varrho| \geq \frac{c_2}{\log q},$$

and hence

$$\left| \sum_{\varrho \in \mathcal{L}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| \ll \log q \cdot x^{3/4} = O(E), \quad (4.6)$$

where we have taken into account that $q \leq x^{c_4}$ and that $\frac{1+\beta_0}{2} < \frac{3}{4}$. Combining (4.6) and (4.5) we obtain (4.4) in this case.

To treat the case in which χ is exceptional, suppose now that $L(\beta^*, \chi) = 0$ with β^* being the exceptional zero. Since χ is a primitive character, one can use the functional equation for L -functions to conclude that $L(\delta^*, \chi) = 0$ where $\delta^* = 1 - \beta^*$ (see, e.g., [3, §9, eq.(8)] and [3, §9, eq.(11)] if $\chi(-1) = 1$ and $\chi(-1) = -1$, respectively). Then

$$\left| \sum_{\varrho \in \mathcal{L}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| \leq \frac{x^{\delta^*}}{\delta^*} + \left| \sum_{\varrho \in \mathcal{L} \setminus \{\delta^*\}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right|.$$

Using Lemma 4.2 one concludes as above that

$$\left| \sum_{\varrho \in \mathcal{L} \setminus \{\delta^*\}} \tilde{m}(\varrho) \frac{x^\varrho}{\varrho} \right| = O(E). \quad (4.7)$$

For the remaining term we use the estimate

$$\delta^* = 1 - \beta^* > Cq^{-\frac{1}{2}} \begin{cases} 1 & \text{if } \chi(-1) = -1, \\ \log q & \text{if } \chi(-1) = 1, \end{cases}$$

with some $C > 0$ (see, e.g., [7]), which yields for any fixed $\varepsilon > 0$:

$$\frac{x^{\delta^*}}{\delta^*} = O(\sqrt{q} \cdot x^\varepsilon) = O(x^{c_4/2} \cdot x^\varepsilon) = O(E).$$

Combining this estimate with (4.7) and (4.5) we also obtain (4.4) in the case that χ is exceptional. \square

Finally, we need the following statement.

Lemma 4.4. *Let \mathcal{U} be a finite set of complex numbers. For any nonzero complex numbers c_u one can find arbitrarily large values of θ for which the function $f(\theta) := \sum_{u \in \mathcal{U}} c_u e^{-u\theta}$ satisfies the lower bound $|f(\theta)| \geq Ce^{-\mu\theta}$, where $\mu := \min_{u \in \mathcal{U}} \{\Re u\}$ and C is a positive constant depending only on f .*

Proof. Replacing $f(\theta)$ with $e^{u_0\theta}f(\theta)$, where u_0 denotes any fixed element of \mathcal{U} for which $\Re u_0 = \mu$, we can assume without loss of generality that $\mu = 0$. Moreover, denoting by \mathcal{U}_+ the set of $u \in \mathcal{U}$ with $\Re u > 0$, we clearly have $\sum_{u \in \mathcal{U}_+} c_u e^{-u\theta} = o(1)$ as $\theta \rightarrow \infty$; hence, we can also assume that $\Re u = 0$ for all $u \in \mathcal{U}$. With these assumptions, the lemma is a consequence of Wiener's Lemma:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |f(\theta)|^2 d\theta = \sum_{u \in \mathcal{U}} |c_u|^2. \quad (4.8)$$

Indeed, the premise that $\limsup_{\theta \rightarrow \infty} |f(\theta)| = 0$ leads to

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |f(\theta)|^2 d\theta = \lim_{\theta \rightarrow \infty} |f(\theta)|^2 = 0,$$

which is impossible in view of (4.8); therefore, $\limsup_{\theta \rightarrow \infty} |f(\theta)| > 0$, which completes the proof. \square

4.2 A relation involving δ

Thanks to Hildebrand [9] it is known that for every $\varepsilon > 0$ there is a number $p_0(\varepsilon) \geq 2$ such that

$$\left| \sum_{n \leq X} (n|p) \right| \leq \varepsilon X \quad (p \geq p_0(\varepsilon), X \geq p^{1/4}).$$

The next statement is an immediate consequence of Hildebrand's result.

Lemma 4.5. *The estimate*

$$|\mathcal{N}_p(X)| = (1/2 + o(1))X \quad (p \rightarrow \infty)$$

holds for all $X \geq p^{1/4}$, where \mathcal{N}_p is given by (1.5), and the function implied by $o(1)$ depends only on p .

In what follows, let C be a large positive number. All constants implied by the symbols O and \ll may depend on $\kappa, \lambda, \mathbf{d}, C$ but are absolute otherwise. The symbol $\underline{o}(1)$ in any expression below indicates an error term that tends to zero as p tends to infinity *within the set* \mathcal{P} . Any function of p implied by $\underline{o}(1)$ may depend on $\kappa, \lambda, \mathbf{d}, C$ but is independent of all other parameters.

For every prime $p \in \mathcal{P}$ let \mathcal{K}_p denote the set of squarefree integers $k > 1$ with the property that $(q|p) = -1$ for all primes q dividing k . The next result is based on the inclusion-exclusion principle.

Lemma 4.6. *Uniformly for $\theta \in [0, C]$ we have*

$$|\mathcal{N}_p(p^\theta)| = \sum_{\substack{k \leq p^\theta \\ k \in \mathcal{K}_p}} (-1)^{\omega(k)+1} |\{m \leq p^\theta/k : (m|p) = -(k|p)\}| + \underline{o}(1)p^\theta,$$

where $\omega(k)$ is the number of distinct prime divisors of k .

Proof. For each $p \in \mathcal{P}$ let \mathcal{A}_p denote the set of ordered pairs given by

$$\mathcal{A}_p := \{(m, k) : k \in \mathcal{K}_p, m \leq p^\theta/k, (m|p) = -(k|p)\}.$$

Then

$$\sum_{(m,k) \in \mathcal{A}_p} (-1)^{\omega(k)+1} = \sum_{\substack{k \leq p^\theta \\ k \in \mathcal{K}_p}} (-1)^{\omega(k)+1} |\{m \leq p^\theta/k : (m|p) = -(k|p)\}|. \quad (4.9)$$

Next, split $\mathcal{N}_p(p^\theta)$ into a disjoint union $\mathcal{N}_1 \cup \mathcal{N}_2$, where

$$\mathcal{N}_1 := \{n \in \mathcal{N}_p(p^\theta) : q^2 \nmid n \text{ if } (q|p) = -1\},$$

and $\mathcal{N}_2 := \mathcal{N}_p(p^\theta) \setminus \mathcal{N}_1$. Since $n_0(p) > p^\kappa$ and $\kappa > 0$ we have

$$|\mathcal{N}_2| \leq \sum_{p^\kappa < q \leq p^\theta} |\{n \leq p^\theta : q^2 \mid n \text{ and } (q|p) = -1\}| \leq p^\theta \sum_{q > p^\kappa} q^{-2} = \underline{o}(1)p^\theta,$$

and therefore

$$|\mathcal{N}_p(p^\theta)| = |\mathcal{N}_1| + \underline{o}(1)p^\theta. \quad (4.10)$$

Each number $n \in \mathcal{N}_1$ can be factored as $n^+ n^-$, where

$$n^+ := \prod_{\substack{q^\alpha \parallel n \\ (q|p)=+1}} q^\alpha \quad \text{and} \quad n^- := \prod_{\substack{q \mid n \\ (q|p)=-1}} q.$$

Let $r_j(n)$ denote the number of pairs $(m, k) \in \mathcal{A}_p$ such that $mk = n$ and $\omega(k) = j$. Then

$$r_j(n) = |\{k > 1 : k \mid n^-, \omega(k) = j\}| = \binom{\omega(n^-)}{j}.$$

Hence, denoting by \mathcal{B}_p the subset of \mathcal{A}_p consisting of pairs (m, k) for which $mk \in \mathcal{N}_1$, we have

$$\sum_{(m,k) \in \mathcal{B}_p} (-1)^{\omega(k)+1} = \sum_{n \in \mathcal{N}_1} \sum_{j=1}^{\omega(n^-)} (-1)^{j+1} r_j(n) = |\mathcal{N}_1| \quad (4.11)$$

since each inner sum is

$$\sum_{j=1}^{\omega(n^-)} (-1)^{j+1} \binom{\omega(n^-)}{j} = 1.$$

Finally, we observe that the set $\mathcal{C}_p := \mathcal{A}_p \setminus \mathcal{B}_p$ consists of pairs $(m, k) \in \mathcal{A}_p$ such that $q^2 \mid mk$ for some prime nonresidue q . Fixing $\varepsilon := \kappa/(2C)$ and using the divisor bound $\sum_{k \mid n} 1 \ll n^\varepsilon$ for all $n \in \mathbb{N}$, for any $\theta \in [0, C]$ we derive that

$$\begin{aligned} |\mathcal{C}_p| &\leq \sum_{p^\kappa < q \leq p^\theta} \sum_{\substack{n \leq p^\theta \\ q^2 \mid n}} \sum_{k \mid n} 1 \ll p^{\theta\varepsilon} \sum_{p^\kappa < q \leq p^\theta} \sum_{\substack{n \leq p^\theta \\ q^2 \mid n}} 1 \\ &\leq p^{\theta(1+\varepsilon)} \sum_{q > p^\kappa} q^{-2} \ll p^{\theta(1+\varepsilon)-\kappa} \leq p^{\theta-\kappa/2} = \underline{o}(1)p^\theta. \end{aligned}$$

Using this result together with (4.10) and (4.11) we deduce that

$$\sum_{(m,k) \in \mathcal{A}_p} (-1)^{\omega(k)+1} = \sum_{(m,k) \in \mathcal{B}_p} (-1)^{\omega(k)+1} + \underline{o}(1)p^\theta = |\mathcal{N}_p(p^\theta)| + \underline{o}(1)p^\theta.$$

In view of (4.9), this completes the proof. \square

Next, using (1.7) we see that for fixed $k \in \mathcal{K}_p$ and uniformly for $\vartheta \in [0, C]$ we have

$$|\{n \leq p^\vartheta : (n|p) = -(k|p)\}| = (\delta_{k,p}(\vartheta) + \underline{o}(1))p^\vartheta, \quad (4.12)$$

where

$$\delta_{k,p} := \begin{cases} \delta & \text{if } (k|p) = +1, \\ 1 - \delta & \text{if } (k|p) = -1. \end{cases}$$

Hence, from Lemma 4.6 we deduce the estimate

$$|\mathcal{N}_p(p^\theta)| = \sum_{\substack{k \leq p^\theta \\ k \in \mathcal{K}_p}} (-1)^{\omega(k)+1} \left(\delta_{k,p} \left(\theta - \frac{\log k}{\log p} \right) + \underline{o}(1) \right) p^\theta k^{-1} + \underline{o}(1)p^\theta.$$

Mertens' theorem yields the bound

$$\sum_{\substack{k \leq p^\theta \\ k \in \mathcal{K}_p}} k^{-1} \leq \left(\sum_{p^\kappa < q \leq p^\theta} q^{-1} \right)^{\lfloor \theta/\kappa \rfloor} \leq (\log(C/\kappa) + O(1))^{C/\kappa} = O(1), \quad (4.13)$$

where $\lfloor \cdot \rfloor$ is the floor function, and therefore

$$|\mathcal{N}_p(p^\theta)| = p^\theta \sum_{\substack{k \leq p^\theta \\ k \in \mathcal{K}_p}} (-1)^{\omega(k)+1} \delta_{k,p} \left(\theta - \frac{\log k}{\log p} \right) k^{-1} + o(1)p^\theta.$$

Dividing both sides by p^θ , using (4.12), and taking into account the fact that $\delta(\theta - u) = 0$ for $u \geq \theta - \kappa$, we derive the relation

$$\delta(\theta) = \sum_{\substack{k \leq p^\theta \\ k \in \mathcal{K}_p \\ \omega(k) \text{ odd}}} k^{-1} - \sum_{\substack{k \leq p^{\theta-\kappa} \\ k \in \mathcal{K}_p}} \delta \left(\theta - \frac{\log k}{\log p} \right) k^{-1} + o(1), \quad (4.14)$$

which holds uniformly for all $\theta \in [0, C]$.

4.3 The functions $\{S_{p,j}\}$

Next, we study the functions defined by

$$S_{p,j}(\theta) := \sum_{\substack{k \leq p^\theta, \, k \in \mathcal{K}_p \\ \omega(k)=j}} k^{-1} \quad (p \in \mathcal{P}, \, j \geq 0, \, \theta \geq 0). \quad (4.15)$$

Each function $S_{p,j}$ is a nondecreasing step function of bounded variation on any finite interval. From (4.13) we see that the bound

$$S_{p,j}(\theta) = O(1) \quad (4.16)$$

holds uniformly for $p \in \mathcal{P}$, $j \geq 0$ and $\theta \in [0, C]$. Note that for $j \in \mathbb{N}$ we have

$$S_{p,j}(\theta) = 0 \quad (\theta \in [0, j\kappa]) \quad (4.17)$$

since every integer k occurring in the sum (4.15) has j distinct prime factors, each of size at least $n_0(p) > p^\kappa$.

Lemma 4.7. *Uniformly for $\theta \in [\kappa, C]$ we have*

$$S_{p,1}(\theta) = \delta(\theta) - \sum_{\text{odd } j \geq 3} S_{p,j}(\theta) + \frac{1}{2} \sum_{j \in \mathbb{N}} \int_0^{\theta-\kappa} S_{p,j}(u) \, \mathbf{d}(\theta-u) \, du + \underline{o}(1). \quad (4.18)$$

Proof. Since $\delta(\theta-u) = 0$ for $u \in [\theta-\kappa, \theta]$, using (1.8) we have for all $j \in \mathbb{N}$:

$$\begin{aligned} \sum_{\substack{k \leq p^{\theta-\kappa}, \, k \in \mathcal{K}_p \\ \omega(k)=j}} \delta\left(\theta - \frac{\log k}{\log p}\right) k^{-1} &= \int_0^{\theta-\kappa} \delta(\theta-u) \, dS_{p,j}(u) \\ &= - \int_0^{\theta-\kappa} S_{p,j}(u) \, d\delta(\theta-u) = \frac{1}{2} \int_0^{\theta-\kappa} S_{p,j}(u) \, \mathbf{d}(\theta-u) \, du, \end{aligned}$$

where the integrals are of Riemann-Stieltjes type (note that these integrals are well-defined since $S_{p,j}$ is of bounded variation). The result now follows by inserting this expression into (4.14). \square

Remark. Using (4.17) one sees that both sums in (4.18) have only finitely many nonzero terms, the number of such terms being bounded by a constant that depends only on $\kappa, \lambda, \mathbf{d}, C$.

Lemma 4.8. *Uniformly for $j \in \mathbb{N}$ and $\theta \in [0, C]$ we have*

$$S_{p,j}(\theta) = \frac{1}{j} \sum_{\substack{q \leq p^\theta \\ (q|p)=-1}} S_{p,j-1}\left(\theta - \frac{\log q}{\log p}\right) q^{-1} + \underline{o}(1).$$

Proof. For any natural number k , let $\omega(k)$ be the number of distinct prime divisors of k , and let $\Omega(k)$ be the total number of primes dividing k , counted with multiplicity.

We first show that for $j \in \mathbb{N}$ and $\theta \in [0, C]$ the estimate

$$S_{p,j}(\theta) = \frac{1}{j!} \sum_{\substack{(q_1, \dots, q_j) \\ (q_i|p)=-1 \, \forall i \\ q_1 \cdots q_j \leq p^\theta}} \frac{1}{q_1 \cdots q_j} + \underline{o}(1) \quad (4.19)$$

holds uniformly. We can assume that $j \leq C/\kappa$, for otherwise the sum is empty and thus (4.19) follows immediately from (4.17).

For each $p \in \mathcal{P}$ let \mathcal{L}_p be the set of integers $k > 1$ such that $(q|p) = -1$ for every prime divisor q of k ; note that a number $k \in \mathcal{L}_p$ lies in \mathcal{K}_p if and only if k is squarefree. Let $r_j(k)$ be the number of ordered j -tuples (q_1, \dots, q_j) of primes such that $q_1 \cdots q_j = k$ and $(q_i|p) = -1$ for each i . Then

- $0 \leq r_j(k) \leq j!$;
- $r_j(k) \neq 0$ if and only if $k \in \mathcal{L}_p$ and $\Omega(k) = j$;
- $r_j(k) = j!$ if and only if $k \in \mathcal{K}_p$ and $\omega(k) = j$.

These properties imply that

$$\sum_{\substack{(q_1, \dots, q_j) \\ (q_i|p) = -1 \ \forall i \\ q_1 \cdots q_j \leq p^\theta}} \frac{1}{q_1 \cdots q_j} = j! \sum_{\substack{k \leq p^\theta, \ k \in \mathcal{K}_p \\ \omega(k) = j}} k^{-1} + \sum_{\substack{k \leq p^\theta, \ k \in \mathcal{L}_p \setminus \mathcal{K}_p \\ \Omega(k) = j}} r_j(k) k^{-1}.$$

Dividing both sides by $j!$ and recalling that $j \leq C/\kappa$, we see that

$$\begin{aligned} \left| S_{p,j}(\theta) - \frac{1}{j!} \sum_{\substack{(q_1, \dots, q_j) \\ (q_i|p) = -1 \ \forall i \\ q_1 \cdots q_j \leq p^\theta}} \frac{1}{q_1 \cdots q_j} \right| &\leq \lfloor C/\kappa \rfloor! \sum_{\substack{k \leq p^\theta \\ k \in \mathcal{L}_p \setminus \mathcal{K}_p}} k^{-1} \ll \sum_{p^\kappa < q \leq p^\theta} \sum_{\substack{n \leq p^\theta \\ q^2 | n}} n^{-1} \\ &\ll \sum_{q > p^\kappa} q^{-2} \log(p^\theta) \ll p^{-\kappa} \log(p^\theta) = \underline{o}(1), \end{aligned}$$

which yields (4.19).

To complete the proof, we suppose that $\theta \in [0, C]$ and apply (4.19) with

both j and $j - 1$ to derive that

$$\begin{aligned}
S_{p,j}(\theta) &= \frac{1}{j!} \sum_{\substack{(q_1, \dots, q_j) \\ (q_i|p)=-1 \ \forall i \\ q_1 \cdots q_j \leq p^\theta}} \frac{1}{q_1 \cdots q_j} + \underline{o}(1) \\
&= \frac{1}{j!} \sum_{\substack{q_1 \leq p^\theta \\ (q_1|p)=-1}} \frac{1}{q_1} \sum_{\substack{(q_2, \dots, q_j) \\ (q_i|p)=-1 \ \forall i \\ q_2 \cdots q_j \leq p^\theta / q_1}} \frac{1}{q_2 \cdots q_j} + \underline{o}(1) \\
&= \frac{1}{j} \sum_{\substack{q \leq p^\theta \\ (q|p)=-1}} \left(S_{p,j-1}(\theta - \frac{\log q}{\log p}) + \underline{o}(1) \right) q^{-1} + \underline{o}(1) \\
&= \frac{1}{j} \sum_{\substack{q \leq p^\theta \\ (q|p)=-1}} S_{p,j-1}(\theta - \frac{\log q}{\log p}) q^{-1} + \underline{o}(1),
\end{aligned}$$

where we have used Mertens' theorem in the final step. \square

4.4 The functions $\{S_j\}$

Proposition 4.9. *We have*

(i) *The limits*

$$S_j(\theta) := \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{p,j}(\theta) \quad (j \geq 0, \theta \in [0, C])$$

exist, and the estimate $S_j(\theta) = S_{p,j}(\theta) + \underline{o}(1)$ holds uniformly for $j \geq 0$ and $\theta \in [0, C]$;

(ii) *Each function S_j is continuous at $\theta \in [0, C]$.*

Proof. For $j = 0$ there is nothing to prove, so we assume that $j \in \mathbb{N}$ in what follows. For each $\ell \in \mathbb{N}$ let \mathcal{I}_ℓ be the interval $[0, C_\ell]$, where $C_\ell := \ell\kappa$. By induction on ℓ we show that both statements hold when $C = C_\ell$, the case $\ell = 1$ being an immediate consequence of (4.17).

Now suppose that (i) and (ii) hold with $\ell \in \mathbb{N}$ and $C = C_\ell$, and let $j \in \mathbb{N}$ and $\theta \in \mathcal{I}_{\ell+1}$ be fixed. If $\theta < j\kappa$, then $S_{p,j}(\theta) = S_j(\theta) = 0$ for all $p \in \mathcal{P}$ by (4.17); thus, we can assume without loss of generality that $\theta \geq j\kappa$.

First, consider the case that $j \geq 2$. As $n_0(p) > p^\kappa$, Lemma 4.8 implies that

$$S_{p,j}(\theta) = \frac{1}{j} \sum_{\substack{p^\kappa < q \leq p^\theta \\ (q|p)=-1}} q^{-1} S_{p,j-1}\left(\theta - \frac{\log q}{\log p}\right) + \underline{o}(1).$$

For any prime $q \in (p^\kappa, p^\theta]$ we have $\theta - \frac{\log q}{\log p} \in \mathcal{I}_\ell$; therefore, using (i) with $C = C_\ell$ together with (4.16) and (4.17) we derive that

$$\begin{aligned} S_{p,j}(\theta) &= \frac{1}{j} \sum_{\substack{p^\kappa < q \leq p^\theta \\ (q|p)=-1}} q^{-1} \left(S_{j-1}\left(\theta - \frac{\log q}{\log p}\right) + \underline{o}(1) \right) + \underline{o}(1) \\ &= \frac{1}{j} \sum_{\substack{p^\kappa < q \leq p^\theta \\ (q|p)=-1}} q^{-1} S_{j-1}\left(\theta - \frac{\log q}{\log p}\right) + \underline{o}(1) \\ &= \frac{1}{j} \sum_{\substack{p^\kappa < q \leq p^{\theta-(j-1)\kappa} \\ (q|p)=-1}} q^{-1} S_{j-1}\left(\theta - \frac{\log q}{\log p}\right) + \underline{o}(1), \end{aligned}$$

where we have used (4.17) with $j-1$ in place of j to reduce the range of q in the last sum. By (ii) with $C = C_\ell$, the function S_{j-1} is continuous on \mathcal{I}_ℓ , and therefore the last sum can be expressed as a Riemann-Stieltjes integral:

$$\int_{\kappa}^{\theta-(j-1)\kappa} S_{j-1}(\theta - u) dS_{p,1}(u) = - \int_{\kappa}^{\theta-(j-1)\kappa} S_{p,1}(u) dS_{j-1}(\theta - u).$$

Since $j \geq 2$ and $\theta \leq C_{\ell+1}$, one verifies that $u \in \mathcal{I}_\ell$ for all values of u in these integrals; hence, using (ii) with $C = C_\ell$ along with (4.16) and (4.17) we have

$$\begin{aligned} \int_{\kappa}^{\theta-(j-1)\kappa} S_{p,1}(u) dS_{j-1}(\theta - u) &= \int_{\kappa}^{\theta-(j-1)\kappa} (S_1(u) + \underline{o}(1)) dS_{j-1}(\theta - u) \\ &= \int_{\kappa}^{\theta-(j-1)\kappa} S_1(u) dS_{j-1}(\theta - u) + \underline{o}(1) \\ &= - \int_{\kappa}^{\theta-(j-1)\kappa} S_{j-1}(\theta - u) dS_1(u) + \underline{o}(1). \end{aligned}$$

Putting everything together, we have shown that

$$S_{p,j}(\theta) = \frac{1}{j} \int_{\kappa}^{\theta-(j-1)\kappa} S_{j-1}(\theta - t) dS_1(t) + \underline{o}(1).$$

This proves (i) for $C = C_{\ell+1}$ in the case that $j \geq 2$. Considering separately the cases $\theta \leq j\kappa$ and $\theta > j\kappa$, we have established the following relation:

$$S_j(\theta) = \frac{1}{j} \int_0^\theta S_{j-1}(\theta - t) dS_1(t) \quad (j \geq 2, \theta \in \mathcal{I}_{\ell+1}).$$

Next, we prove (ii) for $j \geq 2$ and $\theta \in \mathcal{I}_{\ell+1}$. Let $\varepsilon > 0$ be given, and suppose that $p \in \mathcal{P}$ is large enough so that

$$p^{-\kappa} < \varepsilon/3 \quad \text{and} \quad |S_j(\theta) - S_{p,j}(\theta)| < \varepsilon/3.$$

Each $S_{p,j}$ is a step function, and the size of the step at an integer $k > n_0(p)$ is $k^{-1} < p^{-\kappa} < \varepsilon/3$. Since the steps occur on a discrete subset of the real line, it follows that the bound

$$|S_{p,j}(\theta) - S_{p,j}(\theta')| < \varepsilon/3$$

holds for all θ' in a small neighborhood of θ . Hence, if $\theta' \in \mathcal{I}_{\ell+1}$ is sufficiently close to θ , then

$$|S_j(\theta) - S_j(\theta')| \leq |S_j(\theta) - S_{p,j}(\theta)| + |S_{p,j}(\theta) - S_{p,j}(\theta')| + |S_{p,j}(\theta') - S_j(\theta')| < \varepsilon.$$

Therefore, S_j is continuous at θ .

It remains to verify (i) and (ii) for the case $j = 1$. Since $\theta \geq \kappa$, we can apply Lemma 4.7; in view of the remark that follows Lemma 4.7, both sums in (4.18) have at most finitely many nonzero terms, hence we derive that

$$S_{p,1}(\theta) = \delta(\theta) - \sum_{\text{odd } j \geq 3} S_j(\theta) + \frac{1}{2} \sum_{j \in \mathbb{N}} \int_0^{\theta - \kappa} S_j(u) d(\theta - u) du + \underline{o}(1),$$

which implies (i) for $C = C_{\ell+1}$. This relation can also be used to prove (ii), or one can use an argument identical to the one given above for the case $j \geq 2$. This completes the induction, and the proposition is proved. \square

The next corollary follows immediately from the statement and proof of Proposition 4.9; we omit the details.

Corollary 4.10. *For any $j \in \mathbb{N}$ the limit*

$$S_j(\theta) := \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{p,j}(\theta)$$

exists and is finite for all $\theta \geq 0$, the function S_j is continuous on $[0, \infty)$, and the following relations hold for all $\theta \geq 0$:

$$S_j(\theta) = \frac{1}{j} \int_0^\theta S_{j-1}(\theta - u) dS_1(u), \quad (4.20)$$

$$\delta(\theta) = \sum_{\text{odd } j \in \mathbb{N}} S_j(\theta) - \frac{1}{2} \sum_{j \in \mathbb{N}} \int_0^\theta S_j(u) d(\theta - u) du. \quad (4.21)$$

4.5 Laplace transform

In what follows, we use σ and t to denote the real and imaginary parts of the complex number s , respectively.

For any locally integrable function f on $[0, \infty)$, we denote by $\mathcal{L}_s(f)$ the Laplace transform of f , i.e.,

$$\mathcal{L}_s(f) := \int_0^\infty e^{-s\theta} f(\theta) d\theta.$$

Here, s is a complex variable for which the integral converges absolutely. For an excellent account of the theory of the Laplace transform, the reader is referred Widder [21].

Lemma 4.11. *The Laplace integrals $\mathcal{L}_s(\mathbf{d})$, $\mathcal{L}_s(\delta)$ and $\{\mathcal{L}_s(S_j) : j \in \mathbb{N}\}$ converge absolutely in the region $\{s \in \mathbb{C} : \sigma > 0\}$.*

Proof. Since \mathbf{d} and δ are bounded on $[0, \infty)$, the integrals $\mathcal{L}_s(\mathbf{d})$ and $\mathcal{L}_s(\delta)$ converge when $\sigma > 0$.

With Mertens' theorem we can bound

$$S_{p,j}(\theta) \leq \left(\sum_{p^\kappa < q \leq p^\theta} q^{-1} \right)^j \leq (\log(\theta/\kappa) + O((\log p)^{-1}))^j \quad (q \in \mathcal{P}).$$

Letting $p \rightarrow \infty$ we derive the bound

$$S_j(\theta) \leq (\log(\theta/\kappa))^j \quad (\theta \geq \kappa), \quad (4.22)$$

and it follows that the integrals $\{\mathcal{L}_s(S_j) : j \in \mathbb{N}\}$ converge when $\sigma > 0$. \square

Lemma 4.12. *There is a constant $C > 0$ that depends only on $\kappa, \lambda, \mathbf{d}$ such that the inequality*

$$\max \{ |s \mathcal{L}_s(S_1)|, |\mathcal{L}_s(\mathbf{d})| \} < 1 \quad (4.23)$$

holds everywhere in the region

$$\mathcal{R} := \{s \in \mathbb{C} : \sigma > C, |t| < \tfrac{1}{2}\kappa^{1/2}\sigma^{3/2}\}, \quad (4.24)$$

and we have

$$\mathcal{L}_s(S_1) = \frac{1}{2s} \sum_{n \in \mathbb{N}} \frac{\mathcal{L}_s(\mathbf{d})^n}{n} \quad (s \in \mathcal{R}). \quad (4.25)$$

Proof. Fix $s = \sigma + it$ in the region (4.24), and note that

$$|s|^2 = \sigma^2 + t^2 < \sigma^2(1 + \tfrac{1}{4}\kappa\sigma) \leq \sigma^2 e^{\kappa\sigma/4}. \quad (4.26)$$

Using (4.22) with $j = 1$ we have

$$e^{-\sigma\theta} |S_1(\theta)| \leq e^{-\sigma\theta} \log(\theta/\kappa) \leq C_1 e^{-\sigma\theta/2} \quad (\theta \geq \kappa)$$

for some constant C_1 that depends only on $\kappa, \lambda, \mathbf{d}$. Taking into account that S_1 vanishes on $[0, \kappa]$ we have

$$|\mathcal{L}_s(S_1)| \leq C_1 \int_{\kappa}^{\infty} e^{-\sigma\theta/2} d\theta = 2C_1 \sigma^{-1} e^{-\kappa\sigma/2},$$

which together with (4.26) yields the bound

$$|s \mathcal{L}_s(S_1)|^2 \leq 4C_1^2 e^{-3\kappa\sigma/4} < 4C_1^2 e^{-3\kappa C/4}.$$

Hence, if $C > 2\kappa^{-1} \log(4C_1^2)$, then $|s \mathcal{L}_s(S_1)| < 1$.

Replacing C by a larger constant, if necessary, the same method shows that $|s \mathcal{L}_s(\delta)| < 1$, using the bound $\delta(\theta) \leq \frac{1}{2}$ for all $\theta \geq \kappa$ instead of (4.22). In view of the fact that $s \mathcal{L}_s(\delta) = \frac{1}{2} \mathcal{L}_s(\mathbf{d})$, the first statement is proved.

From (4.20) it follows that

$$\mathcal{L}_s(S_j) = \frac{s \mathcal{L}(S_1) \mathcal{L}_s(S_{j-1})}{j} \quad (j \in \mathbb{N}).$$

By induction on j this leads to the relations

$$\mathcal{L}_s(S_j) = \frac{s^{j-1} \mathcal{L}_s(S_1)^j}{j!} \quad (j \in \mathbb{N}).$$

From (4.21) we further deduce that

$$\begin{aligned}\mathcal{L}_s(\delta) &= \sum_{\text{odd } j \in \mathbb{N}} \mathcal{L}_s(S_j) - \frac{1}{2} \sum_{j \in \mathbb{N}} \mathcal{L}_s(S_j * \mathbf{d}) \\ &= \sum_{\text{odd } j \in \mathbb{N}} \frac{s^{j-1} \mathcal{L}_s(S_1)^j}{j!} - \frac{\mathcal{L}_s(\mathbf{d})}{2} \sum_{j \in \mathbb{N}} \frac{s^{j-1} \mathcal{L}_s(S_1)^j}{j!}.\end{aligned}$$

Note that the sums converge absolutely by (4.23). From the previous relation it follows that

$$\frac{\mathcal{L}_s(\mathbf{d})}{2} = s \mathcal{L}_s(\delta) = \frac{1}{2} (e^{s \mathcal{L}_s(S_1)} - e^{-s \mathcal{L}_s(S_1)}) - \frac{\mathcal{L}_s(\mathbf{d})}{2} (e^{s \mathcal{L}_s(S_1)} - 1),$$

which leads to

$$\mathcal{L}_s(S_1) = -\frac{1}{2s} \log(1 - \mathcal{L}_s(\mathbf{d})).$$

Using (4.23) and the Maclaurin series for $\log(1 - u)$ we obtain (4.25). \square

Proposition 4.13. *For $\theta \geq 0$ we have*

$$S_1(\theta) = \sum_{n \in \mathbb{N}} \frac{(\delta * \mathbf{d}^{*(n-1)})(\theta)}{n} \quad (\theta \geq 0). \quad (4.27)$$

Proof. Let T_1 be the function of θ defined by the right side of (4.27). As δ is continuous on $[0, \infty)$, the same is true of $\delta * \mathbf{d}^{*(n-1)}$ for each n . Since $\mathbf{d}^{*(n-1)}$ vanishes for $\theta \leq (n-1)\kappa$, the same is also true for $\delta * \mathbf{d}^{*(n-1)}$; this implies that T_1 is the sum of finitely many continuous functions on any compact interval in $[0, \infty)$, and thus T_1 is continuous on all of $[0, \infty)$. Since

$$\mathcal{L}_s(\delta * \mathbf{d}^{*(n-1)}) = \mathcal{L}_s(\delta) \mathcal{L}_s(\mathbf{d})^{n-1} = \frac{\mathcal{L}_s(\mathbf{d})^n}{2s}$$

for all $n \in \mathbb{N}$ and all $s \in \mathbb{C}$ with $\sigma > 0$, we have

$$\mathcal{L}_s(T_1) = \sum_{n \in \mathbb{N}} \frac{\mathcal{L}_s(\delta * \mathbf{d}^{*(n-1)})}{n} = \frac{1}{2s} \sum_{n \in \mathbb{N}} \frac{\mathcal{L}_s(\mathbf{d})^n}{n} = \mathcal{L}_s(S_1) \quad (s \in \mathcal{R}),$$

where \mathcal{R} is the region (4.24). Now S_1 and T_1 have the same Laplace transform on \mathcal{R} , hence $S_1(\theta) = T_1(\theta)$ for all $\theta \geq 0$ except possibly on a set of Lebesgue measure zero (see, for example, Widder [21, Theorem 6.3]); as both functions are continuous, we find that $S_1 = T_1$ on $[0, \infty)$, and the proposition has been proved. \square

Corollary 4.14. *The function S_1 is continuously differentiable on (λ, ∞) , and*

$$S_1'(\theta) = \frac{1}{2} \sum_{n \in \mathbb{N}} \frac{\mathbf{d}^{*n}(\theta)}{n} \quad (\theta > \lambda).$$

Proof. Since $\mathbf{d}^{*(n-1)}$ vanishes for $\theta \leq (n-1)\kappa$, for any constant $C > 0$ the relation (4.27) implies that

$$S_1(\theta) = \sum_{n < 1+C/\kappa} \frac{(\delta * \mathbf{d}^{*(n-1)})(\theta)}{n} \quad (0 \leq \theta < C). \quad (4.28)$$

As δ is constant (hence differentiable) on (λ, ∞) , it follows that the function S_1 is differentiable on (λ, C) ; taking $C \rightarrow \infty$ we obtain the first statement of the corollary. The second statement follows from (4.28) using the relation $\delta' = \frac{1}{2}\mathbf{d}$ and well known properties of the Laplace integral; we omit the details. \square

4.6 Two expressions for $S_1'(\theta)$

Combining Theorem 1.1 and Corollary 4.14 we obtain the following statement.

Proposition 4.15. *Let $f(k) := \widehat{\mathbf{d}}(k) - 1$, and let $c > 0$ be a real number such that f does not vanish on the line $\{k \in \mathbb{C} : \Im k = -c\}$. Then*

$$S_1'(\theta) = \frac{1}{2\theta} \left(1 + \sum_{k \in \mathcal{K}} m(k) e^{-ik\theta} + E(c, \theta) e^{-c\theta} \right) \quad (\theta > \lambda),$$

where \mathcal{K} is the set consisting of the (finitely many) zeros k of f which lie in the strip $\Pi_c := \{k \in \mathbb{C} : -c < \Im k < 0\}$, $m(k)$ is the multiplicity of any such zero, and

$$E(c, \theta) := \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{\widehat{\mathbf{d}}'(u - ic)}{1 - \widehat{\mathbf{d}}(u - ic)} e^{-iu\theta} - \widehat{\mathbf{d}}'(u - ic) \right) du.$$

To obtain a second expression for $S_1'(\theta)$, we start with the definition (1.11) and observe that

$$S_1'(\theta) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} (S_1(\theta + \varepsilon) - S_1(\theta)) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{p^\theta < q \leq p^{\theta+\varepsilon} \\ (q|p) = -1}} q^{-1} \quad (4.29)$$

holds for any $\theta > \lambda$. We note that

$$\sum_{\substack{p^\theta < q \leq p^{\theta+\varepsilon} \\ (q|p)=-1}} q^{-1} = \frac{1}{2} \sum_{p^\theta < q \leq p^{\theta+\varepsilon}} q^{-1} - \frac{1}{2} \sum_{p^\theta < q \leq p^{\theta+\varepsilon}} (q|p) q^{-1} - \begin{cases} \frac{1}{2} p^{-1} & \text{if } 1 \in (\theta, \theta + \varepsilon], \\ 0 & \text{otherwise,} \end{cases}$$

and using standard techniques derive the estimates

$$\sum_{p^\theta < q \leq p^{\theta+\varepsilon}} q^{-1} = \varepsilon \theta^{-1} + O((\theta \log p)^{-1} + \varepsilon^2 \theta^{-2})$$

and

$$\sum_{p^\theta < q \leq p^{\theta+\varepsilon}} (q|p) q^{-1} = \sum_{p^\theta < n \leq p^{\theta+\varepsilon}} \frac{\Lambda(n)(n|p)}{n \log n} + O(p^{-\theta});$$

hence from (4.29) it follows that

$$S'_1(\theta) = \frac{1}{2\theta} - \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{p^\theta < n \leq p^{\theta+\varepsilon}} \frac{\Lambda(n)(n|p)}{n \log n}. \quad (4.30)$$

Next, let

$$\psi_p(x) := \sum_{n \leq x} \Lambda(n)(n|p) \quad (x > 0).$$

Using the trivial bound $\psi_p(x) \leq \sum_{n \leq x} \Lambda(n) \ll x$ one verifies that

$$\sum_{p^\theta < n \leq p^{\theta+\varepsilon}} \frac{\Lambda(n)(n|p)}{n \log n} = \int_{p^\theta}^{p^{\theta+\varepsilon}} \frac{d\psi_p(u)}{u \log u} = \int_{p^\theta}^{p^{\theta+\varepsilon}} \frac{\psi_p(u) du}{u^2 \log u} + o(1) \quad (p \rightarrow \infty).$$

Thus, after the change of variables $u \mapsto p^t$ the relation (4.30) transforms to

$$S'_1(\theta) = \frac{1}{2\theta} - \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \frac{\psi_p(p^t) dt}{tp^t}. \quad (4.31)$$

To proceed further, we use the following statement, which is a reformulation of Lemma 4.3 in the special case that χ is the Legendre symbol.

Lemma 4.16. *Let c_4 have the property described in Lemma 4.3. For any constant $c > 0$ one has the representation*

$$\psi_p(p^t) = - \sum_{\varrho \in \mathbb{Z}_p} \frac{\widetilde{m}(\varrho)}{\varrho} p^{t\varrho} + r(p, t, c), \quad (4.32)$$

where the remainder term $r(p, t, c)$ admits the bound

$$r(p, t, c) = O(p^t e^{-ct}) \quad (4.33)$$

uniformly for

$$c_4^{-1} \leq t \leq \frac{1}{K^2} \log p, \quad (4.34)$$

and \mathcal{Z}_p denotes the set of distinct zeros $\varrho = \beta + i\gamma$ of $L(s, (\cdot|p))$ such that $\beta > 1 - 2c/\log p$ and $|\gamma| \leq p$, $\tilde{m}(\varrho)$ is the multiplicity of any such zero, and the implied constant depends only on c . Here $K = K(c)$ is the constant described in Lemma 4.3.

Before proceeding, we study the zeros set \mathcal{Z}_p from Lemma 4.16.

By Lemma 4.1 with $q = T = p$, one observes that

$$|\mathcal{Z}_p| = \sum_{\varrho \in \mathcal{Z}_p} \tilde{m}(\varrho) = N(1 - 2c/\log p, p, (\cdot|p)) \ll \exp(4cc_1), \quad (4.35)$$

which shows that the number of zeros of the L -function $L(s, (\cdot|p))$ in the strip $\{\beta + i\gamma \mid \beta > 1 - 2c/\log p, |\gamma| \leq p\}$ is uniformly bounded with respect to p . Let us define

$$N = \limsup_{p \in \mathcal{P}} |\mathcal{Z}_p|.$$

If $N = 0$, the set \mathcal{Z}_p is empty for $p \in \mathcal{P}$ large enough. If $N > 0$, then without loss of generality (i.e., replacing \mathcal{P} with a suitable infinite subset of \mathcal{P}) we may assume that

$$|\mathcal{Z}_p| = N \quad (p \in \mathcal{P}).$$

After the compactifying the complex plane $\mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$, one can use a straightforward compactness argument to conclude, after possibly replacing \mathcal{P} with a suitable infinite subset of \mathcal{P} , that there are N sequences of zeros $\{\varrho_p^{(n)}\}_{p \in \mathcal{P}}$ of the L -functions $L(\cdot, (\cdot|p))$ such that each sequence is contained in the strip $\{\beta + i\gamma \mid \beta > 1 - 2c/\log p, |\gamma| \leq p\}$, and

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} (\varrho_p^{(n)} - 1) \log p = \ell^{(n)} \in \mathbb{C} \cup \{\infty\} \quad (n = 1, 2, \dots, N). \quad (4.36)$$

Here, the limits are taken with respect to the topology of the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Denote by \mathcal{L} the set of *distinct finite limits* $\ell^{(n)}$ from (4.36); we always set $\mathcal{L} = \emptyset$ if $N = 0$, and it can be the case that $\mathcal{L} = \emptyset$ even if $N > 0$.

Remark. From the definitions it is clear that

$$-2c \leq \Re \ell \leq 0 \quad (\ell \in \mathcal{L}). \quad (4.37)$$

The next lemma provides the second representation for $S'(\theta)$.

Lemma 4.17. *Let \mathcal{L} be the zeros attractor defined above. Then*

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_{\substack{\ell \in \mathcal{L} \\ \Re \ell > -c}} \widehat{m}(\ell) e^{\ell\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty),$$

where the multiplicity $\widehat{m}(\ell)$ of $\ell \in \mathcal{L}$ is given by

$$\widehat{m}(\ell) = \#\{n \mid \ell^{(n)} = \ell\}. \quad (4.38)$$

Proof. If $N = 0$ (and thus, $\mathcal{L} = \emptyset$) we claim that

$$S'_1(\theta) = \frac{1}{2\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty). \quad (4.39)$$

Indeed, the set \mathcal{Z}_p is empty for $p \in \mathcal{P}$ large enough, hence by Lemma 4.16 the function $\psi_p(p^t)$ admits the following uniform estimate

$$\psi_p(p^t) = O(p^t e^{-ct}) \quad (p \rightarrow \infty), \quad (4.40)$$

provided that the parameter t satisfies

$$c_4^{-1} \leq t \leq K^{-2} \log p. \quad (4.41)$$

Recall that by (4.31) we have

$$S'_1(\theta) = \frac{1}{2\theta} - \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \frac{\psi_p(p^t) dt}{tp^t} \quad (\theta > \lambda). \quad (4.42)$$

If, in addition, $\theta > c_4^{-1}$, then for all $t \in [\theta, \theta + \varepsilon]$ the two-sided estimate (4.41) holds provided that ε is fixed and p is large enough. Therefore, taking into account that the double limit in the right hand side of (4.42) exists, one can use the uniform bound (4.40) to conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \frac{\psi_p(p^t) dt}{tp^t} = O(e^{-c\theta}) \quad (\theta \rightarrow \infty), \quad (4.43)$$

which together with (4.42) proves the representation (4.39). This completes the proof of the lemma in the case that $N = 0$.

Next, assume that $N > 0$. Using (4.32) we have

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \left(\sum_{\varrho \in \mathcal{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{qt} - r(p, t, c) \right) \frac{dt}{tp^t} \quad (4.44)$$

for all $\theta > \lambda$, where

$$r(p, t, c) = \psi_p(p^t) + \sum_{\varrho \in \mathcal{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{qt}.$$

Suppose for the moment that we have shown that the limit

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \sum_{\varrho \in \mathcal{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{qt} \frac{dt}{tp^t} \quad (4.45)$$

exists. Under this assumption, (4.44) splits as

$$\begin{aligned} S'_1(\theta) &= \frac{1}{2\theta} + \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \sum_{\varrho \in \mathcal{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{qt} \frac{dt}{tp^t}, \\ &\quad - \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} r(p, t, c) \frac{dt}{tp^t}. \end{aligned}$$

Arguing as in the case $N = 0$, and taking into account the bound (4.33), one obtains that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \frac{r(c, p, t) dt}{tp^t} = O(e^{-c\theta}) \quad (\theta \rightarrow \infty); \quad (4.46)$$

therefore,

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \sum_{\varrho \in \mathcal{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{qt} \frac{dt}{tp^t} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty).$$

Thus, to complete the proof we need to show that the double limit (4.45) exists, and to verify that

$$\frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \int_{\theta}^{\theta+\varepsilon} \sum_{\varrho \in \mathcal{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{qt} \frac{dt}{tp^t} = \frac{1}{2\theta} \sum_{\ell \in \mathcal{L}} \widehat{m}(\ell) e^{\ell\theta} \quad (\theta > \lambda).$$

For fixed $p \in \mathcal{P}$ we have

$$\begin{aligned} \int_{\theta}^{\theta+\varepsilon} \sum_{\varrho \in \mathbb{Z}_p} \frac{\tilde{m}(\varrho)}{\varrho} p^{(\varrho-1)t} \frac{dt}{t} &= \sum_{\substack{n \\ \ell^{(n)} < \infty}} \frac{1}{\varrho_p^{(n)}} \int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t} \\ &\quad + \sum_{\substack{n \\ \ell^{(n)} = \infty}} \frac{1}{\varrho_p^{(n)}} \int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t}. \end{aligned}$$

Using the definition (4.36) of $\ell^{(n)}$ we see that

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{n \\ \ell^{(n)} < \infty}} \frac{1}{\varrho_p^{(n)}} \int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t} = \sum_{\substack{n \\ \ell^{(n)} < \infty}} \int_{\theta}^{\theta+\varepsilon} e^{\ell^{(n)}t} \frac{dt}{t},$$

and therefore

$$\begin{aligned} \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{n \\ \ell^{(n)} < \infty}} \frac{1}{\varrho_p^{(n)}} \int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t} \\ = \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \sum_{\substack{n \\ \ell^{(n)} < \infty}} \int_{\theta}^{\theta+\varepsilon} e^{\ell^{(n)}t} \frac{dt}{t} = \frac{1}{2\theta} \sum_{\ell \in \mathcal{L}} m(\ell) e^{\ell\theta}. \end{aligned}$$

Next, we show that

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{n: \ell^{(n)} = \infty} \frac{1}{\varrho_p^{(n)}} \int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t} = 0. \quad (4.47)$$

Indeed, if $\ell^{(n)} = \infty$, from (4.36) and the estimate

$$|1 - \Re \varrho_p^{(n)}| \log p = (1 - \Re \varrho_p^{(n)}) \log p \leq 2c, \quad (4.48)$$

it follows that

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \lim |\omega(n, p)| = \infty, \quad (4.49)$$

where

$$\omega(n, p) = \Im \varrho_p^{(n)} \log p.$$

If $p \in \mathcal{P}$ is large enough and n is such that $\ell^{(n)} = \infty$, integration by parts yields

$$\begin{aligned} \int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t} &= \int_{\theta}^{\theta+\varepsilon} e^{(\Re \varrho_p^{(n)}-1) \log pt} e^{i \omega(n,p)t} \frac{dt}{t} \\ &= \frac{1}{i \omega(n,p)} \cdot e^{(\Re \varrho_p^{(n)}-1) \log pt} e^{i \omega(n,p)t} \frac{1}{t} \Big|_{\theta}^{\theta+\varepsilon} \\ &\quad - \frac{1}{i \omega(n,p)} \int_{\theta}^{\theta+\varepsilon} \frac{d}{dt} \left(\frac{e^{(\Re \varrho_p^{(n)}-1) \log pt}}{t} \right) e^{i \omega(n,p)t} dt, \quad (4.50) \end{aligned}$$

with $\omega(n,p) \neq 0$. Combining (4.48), (4.49) and (4.50) we have

$$\int_{\theta}^{\theta+\varepsilon} p^{(\varrho_p^{(n)}-1)t} \frac{dt}{t} = o(1) \quad (p \rightarrow \infty, p \in \mathcal{P}),$$

provided that $\ell^{(n)} = \infty$. Taking into account the bound

$$\frac{1}{|\varrho_p^{(n)}|} \leq \frac{1}{1 - 2c/\log p} < 2,$$

which holds if p is large enough, (4.47) is proved. Hence,

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_{\ell \in \mathcal{L}} \widehat{m}(\ell) e^{\ell\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty).$$

Finally, the summands with $\Re \ell \leq -c$ can be absorbed by the error term, which completes the proof of the lemma. \square

4.7 Proof of Theorem 1.2

Proof. To prove our main result, Theorem 1.2, we compare two asymptotic representations for the derivative $S'_1(\theta)$ provided by Proposition 4.15

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_{k \in \mathcal{K}} m(k) e^{-ik\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty)$$

and by Lemma 4.17

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_{\ell \in \mathcal{L}: \Re \ell > -c} \widehat{m}(\ell) e^{\ell\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty),$$

respectively.

Comparing those representations yields

$$\sum_{\ell \in \mathcal{L}: \Re \ell > -c} \widehat{m}(\ell) e^{\ell \theta} = \sum_{k \in \mathcal{K}} m(k) e^{-ik\theta} + O(\theta e^{-c\theta}) \quad (\theta \rightarrow \infty). \quad (4.51)$$

However, using Lemma 4.4 and the fact that $\Im k > -c$ for all $k \in \mathcal{K}$, the resulting relation is impossible unless it is the case that $-ik$ lies in the set $\mathcal{L} \cap \{z \in \mathbb{C} \mid -c < \Im(z) \leq 0\}$ and therefore in \mathcal{L} for every $k \in \mathcal{K}$. Since the constant c can be chosen arbitrarily large, this completes the proof of Theorem 1.2. \square

Remark. We note that the upper bound (4.37) from Remark 4.6 can be improved; we have

$$\Re \ell < 0 \quad (\ell \in \mathcal{L}).$$

Indeed, suppose that $\Re \ell = 0$ for some $\ell \in \mathcal{L}$. This means that there is a sequence $(\varrho_p)_{p \in \mathcal{P}}$ with $\varrho_p \in \mathcal{Z}_p$ such that $\Re((\varrho_p - 1) \log p) \rightarrow 0$. By Lemma 4.2 it is clear that each zero ϱ_p of $L(s, (\cdot|p))$ is exceptional if p is large enough; in particular, $\varrho_p = \beta_p$ is a real simple zero. Since $(\beta_p - 1) \log p \rightarrow 0$, the final statement in Lemma 4.2 implies that for all sufficiently large p the set \mathcal{Z}_p consists only of the single zero β_p ; consequently, $\ell = 0$ and thus $\mathcal{L} = \{0\}$. By Lemma 4.17,

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} + O(e^{-c\theta}) = \frac{1}{\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty),$$

which is inconsistent with the asymptotics obtained in Proposition 4.15.

A similar reasoning shows that the set \mathcal{L} is also free of (negative) reals, that is,

$$\mathcal{L} \cap (-\infty, 0) = \emptyset.$$

Indeed, since

$$\widehat{\mathbf{d}}(-i\beta) = \int_{\kappa}^{\lambda} e^{\beta x} \mathbf{d}(x) dx > \int_{\kappa}^{\lambda} \mathbf{d}(x) dx = 1 \quad (\beta > 0),$$

one concludes that the equation $\widehat{\mathbf{d}}(k) = 1$ has no zeros on the negative imaginary axis. Since the roots of the equation $\widehat{\mathbf{d}}(k) = 1$ are complex conjugates of each other, using Proposition 4.15 one observes that the higher order terms in the asymptotic expansion for the derivative $S'_1(\theta)$,

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_{k \in \mathcal{K}} m(k) \cos(\Re(k)\theta) e^{-|\Im(k)|\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty),$$

are oscillatory in any order, which is inconsistent with the asymptotics provided by Lemma 4.17

$$S'_1(\theta) = \frac{1}{2\theta} + \frac{1}{2\theta} \sum_{\substack{\ell \in \mathcal{L} \\ \Re \ell > -c}} \widehat{m}(\ell) e^{\ell\theta} + O(e^{-c\theta}) \quad (\theta \rightarrow \infty),$$

unless $\Im(\ell) \neq 0$ for all $\ell \in \mathcal{L}$.

In particular, the sequence of the L -functions $\{L(s, (\cdot|p))\}_{p \in \mathcal{P}}$ is allowed to have only finitely many terms that have an exceptional zero ϱ_p satisfying the bound

$$\varrho_p > 1 - \frac{c_2}{\log p}$$

from Lemma 4.2.

5 Connection with Heath-Brown's result

The main goal of this section is to show how to deduce the Heath-Brown result concerning the behavior of the Dirichlet L -function from our more general considerations.

We start, however, with preliminary considerations where we discuss the optimality of our main assumptions concerning the distribution of quadratic nonresidues, namely the hypotheses (1.6), (1.7) and (1.8).

Theorem 5.1. *Assume the (κ, λ) hypotheses of Theorem 1.2, i.e., that the conditions of (1.6), (1.7) and (1.8) are met. Then the following inequality*

$$\kappa \leq \lambda/\sqrt{e} \tag{5.1}$$

necessarily holds. Moreover, in the case that

$$\kappa = \lambda/\sqrt{e}, \tag{5.2}$$

the density \mathbf{d} is given by

$$\mathbf{d}(\theta) = \frac{2}{\theta} X_{\kappa, \lambda}(\theta), \tag{5.3}$$

where $X_{\kappa, \lambda}(\cdot)$ is the indicator function of the interval $[\kappa, \lambda]$. Finally, there is an absolute constant $\lambda_0 > 0$ such that if $\lambda < \lambda_0$, then the strict inequality

$$\kappa < \lambda/\sqrt{e} \tag{5.4}$$

necessarily holds.

Proof. Step 1. First, we show that (5.1) holds. Indeed, suppose on the contrary that

$$\kappa > \lambda/\sqrt{e}. \quad (5.5)$$

By Proposition 4.13 we have the representation

$$S_1(\theta) = \sum_{n \in \mathbb{N}} \frac{(\delta * \mathbf{d}^{*(n-1)})(\theta)}{n} \quad (\theta \geq 0).$$

Since both δ and \mathbf{d} vanish on $[0, \kappa)$ by hypothesis, the convolution $\delta * \mathbf{d}^{*(n-1)}$ vanishes on the interval $[0, n\kappa)$ and therefore

$$S_1(\theta) = \delta(\theta) \quad (0 \leq \theta < 2\kappa). \quad (5.6)$$

From (5.5) it follows that $\lambda < 2\kappa$ and thus

$$S_1(\lambda) = \delta(\lambda) = 1/2.$$

Here, the last equality follows from our hypothesis that the distribution \mathbf{d} is supported on $[\kappa, \lambda]$ and that

$$\delta(\lambda) = \frac{1}{2} \int_0^\lambda \mathbf{d}(u) du = \frac{1}{2}.$$

Recalling that

$$S_1(\theta) = \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{q \leq p^\theta \\ (q|p)=-1}} \frac{1}{q} = \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{p^\kappa < q \leq p^\theta \\ (q|p)=-1}} \frac{1}{q}$$

and using Mertens' theorem, one gets the bound

$$S_1(\lambda) = \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \sum_{\substack{p^\kappa < q \leq p^\lambda \\ (q|p)=-1}} \frac{1}{q} \leq \lim_{p \rightarrow \infty} \sum_{p^\kappa < q \leq p^\lambda} \frac{1}{q} = \log \frac{\lambda}{\kappa}.$$

Hence

$$\frac{1}{2} \leq \log \frac{\lambda}{\kappa},$$

which is inconsistent with (5.5). The proof of (5.1) is complete.

Step 2. Next, we show that (5.2) implies (5.3). Using Mertens' theorem again we obtain that

$$\begin{aligned} \frac{1}{2} &= S_1(\lambda) = S_1(\theta) + (S_1(\lambda) - S_1(\theta)) \\ &\leq S_1(\theta) + \log \frac{\lambda}{\theta} \leq \log \frac{\theta}{\kappa} + \log \frac{\lambda}{\theta} = \log \frac{\lambda}{\kappa} = \frac{1}{2}, \end{aligned}$$

and thus

$$S_1(\theta) = \log \frac{\theta}{\kappa} \quad (\kappa \leq \theta \leq \lambda). \quad (5.7)$$

It remains to observe that $\lambda = \kappa\sqrt{e} < 2\kappa$, and therefore one can use (5.6) to conclude that

$$\delta(\theta) = S_1(\theta) = \log \frac{\theta}{\kappa} \quad (\kappa \leq \theta \leq \lambda),$$

which proves (5.3) in view of the equality

$$\delta(\theta) = \frac{1}{2} \int_0^\theta \mathbf{d}(u) \, du \quad (\theta \geq 0). \quad (5.8)$$

Step 3. Finally, we prove the remaining assertion of the theorem for

$$\lambda_0 = \frac{c_2}{4|\Im(k_0)|},$$

where c_2 is the constant from Lemma 4.2, and k_0 is one of the roots of the equation

$$2 \int_{1/(4\sqrt{e})}^{1/4} e^{ikx} \frac{dx}{x} = 1. \quad (5.9)$$

that lie closest to the real axis.

Suppose that inequality (5.4) does not hold; then, by Step 1, we have

$$\kappa = \lambda/\sqrt{e}. \quad (5.10)$$

By Step 2, the probability distribution \mathbf{d} is given by

$$\mathbf{d}(\theta) = \mathbf{d}_\lambda(\theta) = \frac{2}{\theta} X_{\kappa, \lambda}(\theta).$$

Evaluating the Fourier transform of the function d_λ we have

$$\widehat{d}_\lambda(k) := 2 \int_{\lambda/\sqrt{e}}^\lambda e^{ikx} \frac{dx}{x} = 2 \int_{1/(4\sqrt{e})}^{1/4} e^{4i\lambda kx} \frac{dx}{x}.$$

Thus, the zeros of the equation

$$\widehat{d}_\lambda(k) = 1$$

can be obtained from the roots of the equation (5.9) by rescaling $k \rightarrow 4\lambda k$.

By Theorem 1.2, there is a complex sequence ϱ_p with $L(\varrho_p, (\cdot|p)) = 0$ such that

$$(\varrho_p - 1) \log p \rightarrow -ik_0.$$

Since, by hypothesis,

$$\lambda < \lambda_0 = \frac{c_2}{4|\Im(k_0)|},$$

one obtains that

$$1 - \Re(\varrho_p) < 4\lambda \frac{|\Im(k_0)|}{\log p} < 4\lambda_0 \frac{|\Im(k_0)|}{\log p} = \frac{c_2}{\log p} \quad (p \text{ is large enough}).$$

By Lemma 4.2 the roots ϱ_p are exceptional zeros if p is large enough, and hence

$$\mathcal{L} \cap (-\infty, 0) \neq \emptyset,$$

where \mathcal{L} is the zeros attractor defined by (4.36). However, this is impossible in view of Remark 4.7, and this contradiction completes the proof. \square

Remark. It follows from Theorem 5.1 that the inequality (5.1) cannot be relaxed. Thus we see that the hypotheses (1.6), (1.7) and (1.8) together imply that the probability distribution \mathbf{d} cannot be “too concentrated” in a certain sense; in particular, the convex hull of the support of the probability distribution must always contain the critical interval $[\lambda/\sqrt{e}, \lambda]$.

In the case that $\lambda = 1/4$, the exponent

$$\kappa = \frac{1}{4\sqrt{e}}$$

coincides with the exponent in the Burgess bound. In particular, from Theorem 5.1 it follows that if $\lambda = 1/4$ and $\kappa = 1/(4\sqrt{e})$, then

$$\delta(\theta) = \log 4\theta\sqrt{e} \quad (1/(4\sqrt{e}) \leq \theta \leq 1/4),$$

provided that the requirements (1.7) and (1.8) are met.

As it turns out, in the special case that $\kappa = \lambda/\sqrt{e}$, one can replace the hypotheses (1.7) and (1.8) by a much weaker condition (see (5.11) below) and obtain a considerably stronger result, which automatically guarantees the existence of the probability distribution \mathbf{d} of the form (5.3).

Lemma 5.2. *Suppose that $0 < \lambda \leq 1/4$, and let $\kappa = \lambda/\sqrt{e}$. Assume the hypothesis (1.6). Suppose, in addition, that*

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \frac{|\mathcal{N}_p(p^\theta)|}{p^\theta} = \frac{1}{2} \quad (\theta \geq \lambda). \quad (5.11)$$

Then for all $\theta \geq 0$ one has

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \frac{|\mathcal{N}_p(p^\theta)|}{p^\theta} = \delta(\theta), \quad (5.12)$$

where the density function δ has the form

$$\delta(\theta) := \frac{1}{2} \int_0^\theta \mathfrak{d}(u) du \quad (\theta \geq 0) \quad (5.13)$$

with

$$\mathfrak{d}(\theta) = \frac{2}{\theta} \quad (\kappa \leq \theta \leq \lambda).$$

Proof. If $0 \leq \theta \leq \kappa$, then (5.12) is trivial with $\delta(\theta) = 0$ in view of (1.6), whereas if $\theta \geq \lambda$, then (5.12) holds with $\delta(\theta) = 1/2$ by hypothesis.

Now suppose that $\kappa = \lambda/\sqrt{e} < \theta < \lambda$, and let $p \in \mathcal{P}$ be fixed. Since $2\kappa > \theta$ it is clear that a natural number $n \leq p^\theta$ is a nonresidue if and only if $n = qm$ for some prime nonresidue q and natural number m , and in this case the pair (q, m) is determined uniquely by n . Therefore,

$$|\mathcal{N}_p(p^\theta)| = \sum_{\substack{q \leq p^\theta \\ (q|p)=-1}} \left\lfloor \frac{p^\theta}{q} \right\rfloor = S_{p,1}(\theta) p^\theta + O\left(\frac{p^\theta}{\log p}\right), \quad (5.14)$$

where (as before)

$$S_{p,1}(\theta) := \sum_{\substack{q \leq p^\theta \\ (q|p)=-1}} q^{-1}.$$

Note that we have used the Prime Number Theorem and the fact that $\theta \gg 1$ to bound the error term in (5.14).

From (5.11) and (5.14) one concludes that the limit

$$\frac{1}{2} = S_1(\lambda) = \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{1,p}(\lambda)$$

exists. Using Mertens' theorem, we have

$$\begin{aligned} \frac{1}{2} &= S_1(\lambda) = \liminf_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{1,p}(\lambda) + \limsup_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} (S_1(\lambda) - S_{1,p}(\theta)) \\ &\leq \limsup_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{1,p}(\theta) + \log \frac{\lambda}{\theta} \leq \log \frac{\theta}{\kappa} + \log \frac{\lambda}{\theta} = \log \frac{\lambda}{\kappa} = \frac{1}{2}, \end{aligned}$$

and hence

$$S_1(\theta) = \liminf_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{1,p}(\theta) = \limsup_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{1,p}(\theta) = \lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} S_{1,p}(\theta) = \log \frac{\theta}{\kappa}$$

for all $\theta \in (\kappa, \lambda)$. In turn, by (5.14) one gets that

$$\lim_{\substack{p \rightarrow \infty \\ p \in \mathcal{P}}} \frac{|\mathcal{N}_p(p^\theta)|}{p^\theta} = \delta(\theta) \quad (\kappa < \theta < \lambda),$$

thus (5.12) holds in the full range of $\theta \geq 0$. \square

As a corollary of Theorem 1.2, we are in a position to give an independent proof the following result, which is originally due to Heath-Brown (for a more precise statement, see Diamond *et al* [4, Appendix], where a reconstruction of Heath-Brown's work is given).

Corollary 5.3. *Suppose that*

$$(n|p) = 1 \quad (1 \leq n \leq p^{1/(4\sqrt{e})}) \quad (5.15)$$

for all primes p in some infinite set \mathcal{P} . Then for every zero z of the function

$$H(z) := \frac{2}{z} \int_{1/\sqrt{e}}^1 (1 - e^{-zu}) \frac{du}{u},$$

there is a sequence $(\varrho_p)_{p \in \mathcal{P}}$ such that each term ϱ_p is a zero of the L -function $L(s, (\cdot|p))$, and

$$(1 - \varrho_p) \log p = -4z + \underline{\varrho}(1). \quad (5.16)$$

Proof. Combining (5.15) with Lemma 4.5, we see that the hypotheses of Lemma 5.2 are met with

$$\kappa = \frac{1}{4\sqrt{e}} \quad \text{and} \quad \lambda = \frac{1}{4}.$$

Therefore, the hypotheses (1.6), (1.7) and (1.8) are met with the function δ and the probability distribution \mathbf{d} given by (1.9) and (1.10), respectively. For the probability distribution \mathbf{d} given by (1.10) one easily verifies that

$$H\left(-\frac{ik}{4}\right) = \frac{4i}{k}(1 - \widehat{\mathbf{d}}(k));$$

therefore, the asymptotic representation (5.16) follows from Theorem 1.2. \square

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